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Alexander Ziwek

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A TREATISE ON FLUXIONS.

IN TWO VOLUMES.

BY

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Late Professor of Mathematics in the University of Edinburgh, and Fellow of the Royal Society.

SECOND EDITION.

TO WHICH IS PREFIXED

AN ACCOUNT OF HIS LIFE.

THE WHOLE CAREFULLY CORRECTED AND REVISED BY

An Eminent Mathematician.

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TREATISE ON FLUXIONS.

BOOK I.

CHAP. XII.

Of the Method of Infinitesimals, of the Limits of Ratios, and of the general Theorems which are derived from this Doctrine for the Resolution of geometrical and philosophical Problems.

495. IN the account which we have given of the method of fluxions, in the preceding part of this treatise, magnitudes were supposed to be generated by motion; and, by comparing the increments that were generated in any equal successive parts of the time, it was first determined whether the motion was uniform, accelerated, or retarded. When the motion was uniform, the fluxion of the magnitude was measured by the increment which it acquired in a given time. When the motion was accelerated, this increment was resolved into two parts; that which alone would have been generated if the motion had not been accelerated, but had continued uniform from the beginning of the time, and that which was generated in consequence of the continual acceleration of the motion during that time. The latter part was rejected, and the former only retained for measuring the motion at the beginning of the time. And in like manner, when the motion was retarded, the quantity, which was found deficient in consequence of this retardation, was supplied; so that the motion at the term proposed was accurately measured, and the *ratio* of the fluxions always accurately represented, In the method of infinitesimals, the element, by which any quantity increases or decreases, is supposed to be infinitely small, and is generally expressed by two or more terms, some of which are infinitely less than the rest, which

being neglected as of no importance, the remaining terms form what is called the *difference* of the proposed quantity. The terms that are neglected in this manner, as infinitely less than the other terms of the element, are the very same which arise in consequence of the acceleration, or retardation, of the generating motion, during the infinitely small time in which the element is generated; so that the remaining terms express the element that would have been produced in that time, if the generating motion had continued uniform. Therefore those *differences* are accurately in the same ratio to each other as the generating motions or fluxions. And hence, though in this method infinitesimal parts of the elements are neglected, the conclusions are accurately true, without even an infinitely small error, and agree precisely with those that are deduced by the method of fluxions. For example, * in prop. 2, when DG (*fig.* 21), the increment of the base AD of the triangle ADE, is supposed to become infinitely little, the trapezium DGHE (the simultaneous increment of the triangle) consists of two parts, the parallelogram EG, and the triangle EIH; the latter of which is infinitely less than the former, their ratio being that of $\frac{1}{2}$ DG to AD. Therefore, according to this method, the part EIH is neglected, and the remaining part, *viz.* the parallelogram EG, is the *difference* of the triangle ADE. Now it was shown above (art. 93), that EG is precisely that part of the increment of the triangle ADE which is generated by the motion with which this triangle flows, and that EIH is the part of the same increment which is generated in consequence of the acceleration of this motion, while the base by flowing uniformly acquires the augment DG, whether DG be supposed finite or infinitely little. In prop. 3, case 1, the increment DELMHG (*fig.* 22) of the rectangle AE consists of the parallelograms EG, EM, and I*b*; the last of which I*b* becomes infinitely less than EG or EM, when DG and LM the increments of the sides are supposed infinitely small; because I*b* is to EG as LM to AL, and to EM as DG to AD; therefore I*b* being neglected, the sum of the parallelograms EG and EM is the *difference* of the rectangle AE:

* The figures cited from Vol. I. are repeated in this Volume in plate 25, opposite to p. 14.

and

and it was shown in art. 102, that the sum of EG and EM is the space that would have been generated by the motion with which the rectangle AE flows continued uniformly, but that I*b* is the part of the increment of the rectangle which is generated in consequence of the acceleration of this motion, in the time that AD and AL by flowing uniformly acquire the augments DG and LM. The same may be observed of all the other propositions wherein the fluxions of quantities are determined above.

496. In general suppose, as in art. 66, that while the point P (*fig.* 220) describes the right line A*a* with an uniform motion, the point M sets out from L with a velocity that is to the constant velocity of P as L*c* to D*g*, and proceeds in the right line E*e* with a motion continually accelerated or retarded, that LS any space described by M is always to DG the space described in the same time by P as L*f* to D*g*, that *cx* is to D*g* as the difference of the velocities of M at S and L to the constant velocity of P, and that LS is always to LC as L*f* to L*c*. Then LS being always expressed by LC $\mp$ CS, it is manifest that (since LC is to DG as L*c* to D*g*, or as the velocity of M at L to the velocity of P) LC is what would have been described by M if its motion had continued uniformly from L, and that CS arises in this expression in consequence of the acceleration or retardation of the motion of the point M while it describes LS. But if LS and DG be supposed infinitely small increments of EL and AD, *cx* will be infinitely less than D*g*; and since *cf* is less than *cx* by what was shown in art. 66, it follows that *cf* will be infinitely less than L*c*, and CS infinitely less than LC. Therefore when the increment LS is supposed infinitely small, and its expression is resolved into two parts LC, and CS, of which the former LC is always in the same ratio to DG (the simultaneous increment of AD while the increments vary, and the latter CS is infinitely less than the former LC, we may conclude that the part CS is that which arises in consequence of the variation of the motion of M while it describes LS, and is therefore to be neglected in measuring the motion of M at L, or the fluxion of the right line EL. Thus the manner of investigating the differences or fluxions of quantities in the method of infinitesimals may be de-

duced from the principles of the method of fluxions demonstrated above. For instead of neglecting CS because it is infinitely less than LC (according to the usual manner of reasoning in that method), we may reject it, because we may thence conclude that it is not produced in consequence of the generating motion at L, but of the subsequent variations of this motion. And it appears why the conclusions in the method of infinitesimals are not to be represented as if they were only near the truth, but are to be held as accurately true.

497. In order to render the application of this method easy, some analogous principles are admitted, as that the infinitely small elements of a curve are right lines, or that a curve is a polygon of an infinite number of sides, which being produced give the tangents of the curve, and by their inclination to each other measure the curvature. This is as if we should suppose that when the base flows uniformly the ordinate flows with a motion which is uniform for every infinitely small part of time, and increases or decreases by infinitely small differences at the end of every such time. But however convenient this principle may be, it must be applied with caution and art on various occasions. It is usual therefore in many cases to resolve the element of the curve into two or more infinitely small right lines; and sometimes it is necessary (if we would avoid error) to resolve it into an infinite number of such right lines, which are infinitesimals of the second order. In general it is a *postulatum* in this method that we may descend to the infinitesimals of any order whatever as we find it necessary, by which means any error that might arise in the application of it may be discovered and corrected by a proper use of this method itself. This will appear by considering some instances wherein it is said to lead us into error.

498 (*Fig.* 221). The most noted of these is taken from the doctrine of pendulums. If we were to consider the circle ABH, whose diameter AH is perpendicular to the horizon, as a polygon of an infinite number of sides, and consequently the infinitely small arch AB as coinciding with its chord, it would seem to follow that the time of a vibration in such an arch ought to be equal to the time of descent in its chord, which is equal to the time of descent in the diameter HA; whereas if the ratio of those times be

be at all assignable, it must be that of the quadrant of a circle to the diameter, as may be shown from art. 408. But it is easy to discover that we are not in this case to argue from infinitesimals of the first order, since if we should suppose the same arch to coincide with its tangent AT, the time of descent in it would be found infinite. This difficulty however cannot be removed (as some others) by resolving the infinitely small arch AB into two infinitely small chords BD and AD, or tangents BC and AC, or into any finite number of such chords or tangents. The time in the tangent BC must be supposed the half of the time in the chord BA, because BC is equal to CA, and when BDA is supposed infinitely small, BC is one half of BA; the time in CA is the half of the time in BC; consequently the time in BC and CA is three fourths of the time in the chord BA, or diameter HA, which is nearer to the true time in the arch BDA, but is not yet equal to it. By supposing the arch BDA to be continually subdivided into more and more equal parts, and the tangents or chords to be drawn at each division, the times in the circumscribed and inscribed figures will continually approach to the time in the arch, and will at length agree with it when the divisions are supposed infinite in number, in the same manner that the circumscribed and inscribed polygons approach to the circumference of the circle, and are said to coincide with it when the number of their sides is supposed infinite. But the time in such an infinitely small arch is briefly determined by considering it as coinciding with the time in the arch of the cycloid of the same curvature, which was determined in art. 408.

499 (*Fig.* 222). When a curve is considered as a polygon of an infinite number of sides, and CE, EH are any two of those sides, if CE produced meet GH the ordinate from H in T, CT is commonly supposed to be the tangent, and HT the subtense of the angle of contact; and if CL, EI parallel to the base meet the ordinates DE, GH in L and I, IT will be equal to LE, and TH equal to the difference of LE and IH which are the first differences of the ordinates; and hence HT the subtense of the angle of contact is often supposed by authors on this method to be equal to the second difference of the ordinates; whereas it

 follows,

follows, from what was shown above, that when the arch is infinitely diminished, the subtense of the angle of contact is equal to the half of the second difference, or second fluxion of the ordinate, only. But it is obvious that there is no reason why the tangent of the curve at E ought to be supposed to coincide with one of those elements CE, EH, rather than the other; and that it ought to be considered in this method as equally inclined to both, or rather as forming with each infinitely small angles that differ from each other by an angle infinitely less than either. Therefore let the tangent tE*t* be supposed equally inclined to EC and EH, and meet BC, GH in t and *t*; then the *second difference* of the ordinate (or the difference of LE and IH) will be equal to Ct+H*t* or 2H*t*, that is to twice the subtense of the angle of contact. They however who consider the subtense of the angle of contact as equal to the *second difference* of the ordinate, compensate this error by supposing that angle in effect to be double of what it is. But whether we suppose CE and EH to be rectilineal or curvilineal elements of the figure, the subtense of the angle of contact ought to be supposed equal to the half of the *second difference* of the ordinate only. See art. 254. If we would compare these subtenses at different distances from the point of contact, it is better then to consider the element of the curve as an infinitely small arch of a circle, unless when the curvature is of those kinds which were described in art. 377 and 378, that are either less or greater than the curvature of any circle. Hence when the ray of curvature is finite, the subtenses of the same angle of contact are in the duplicate ratio of the arches; but in the cases described in those articles they follow other proportions.

500. When the value of a quantity that is required in a philosophical problem becomes in certain particular cases infinitely great, or infinitely little, the solution would not be always just though such magnitudes were admitted. As when it is required, to find by what centripetal force a curve could be described about a fixed point that is either in the curve, or is so situated that a tangent may be drawn from it to the curve, the value of the force is found infinite at the centre of the forces in the former case, and at the point of contact in the latter; yet

yet it is obvious that an infinite force could not inflect the line described by a body that should proceed from either of these points into a curve; because the direction of its motion in either case passes through the centre of the forces, and no force how great soever that tends towards the centre could cause it to change that direction. But it is to be observed that the geometrical magnitude by which the force is measured is no more imaginary in this than in other cases where it becomes infinite; and philosophical problems have limitations that enter not always into the general solution given by geometry.

501. But to insist on no more instances: what we have chiefly in view is to show how these scruples may be obviated, which the brief manner of proceeding in the method of infinitesimals is apt to suggest to such as enter on the higher parts of geometry, after having been accustomed to a more strict and rigid kind of demonstration in the elementary parts. To such it may seem not to be consistent with the perfect accuracy that is required in geometrical demonstration, that in determining the first differences, any part of the element of the variable quantity should be rejected merely because it is infinitely less than the rest, and that the same part should be afterwards employed for determining the second and higher differences, and resolving some of the most important problems. Nor can we suppose that their scruples will be removed, but rather confirmed, when they come to consider what has been advanced by some of the most celebrated writers on this method, who have expressed their sentiments concerning infinitely small quantities in the precisest terms; while some of them deny their reality, and consider them only as incomparably less than finite quantities, in the same manner as a grain of sand is imcomparably less than the whole earth; and others represent them, in all their orders, as no less real than finite quantities. It was with a view to remove any ground there might seem to be given for scruples of this kind, that we followed a less concise method in the preceding chapters of this treatise, and showed in art. 495 and 496, that a satisfactory account may be given for the more brief way of reasoning that is in use in the method of infinitesimals. When we investigate the first differences, we may reject

the infinitesimal parts of the element, not merely because they are infinitely less than the other parts; but because the quantities generated, and their mutual relations depend upon the generating motions (art. 24, 33, 42, 43), and are discovered by them: and because in measuring these motions, at any term of the time, the infinitesimal parts of the element are not to be regarded, since they are not generated in consequence of those motions themselves, but of their variations from that term; as was shown at length in prop. 2, and its corollaries, and in several other parts of the preceding chapters. The same infinitesimal parts of the element however may serve for measuring the acceleration or retardation of those motions from that term, or the powers which may be conceived to accelerate or retard them at that term: and here the infinitely small parts of the element that are of the third order are neglected for a similar reason, being generated only in consequence of the variation of those powers from that term of the time. In this manner we presume some satisfaction may be given to the scrupulous (who may be apt to demur at the usual way of reasoning in this method), while nothing is neglected without accounting for it; and thus the harmony may appear to be more perfect betwixt the method of fluxions and that of infinitesimals.

502. But however safe and convenient this method may be, some will always scruple to admit infinitely little quantities, and infinite orders of infinitesimals, into a science that boasts of the most evident and accurate principles as well as of the most rigid demonstrations; and therefore we chose to establish so extensive and useful a doctrine in the preceding chapters on more unexceptionable *postulata*. In order to avoid such suppositions, Sir *Isaac Newton* considers the simultaneous increments of the flowing quantities as finite, and then investigates the ratio which is the limit of the various proportions which those increments bear to each other, while he supposes them to decrease together till they vanish; which ratio is the same with the ratio of the fluxions by what was shown in art. 66, 67, and 68. In order to discover this limit, he first determines the ratio of the increments in general, and reduces it to the most simple terms, so as that (generally speaking) a part at least of each term may be independent of the

value

value of the increments themselves; then by supposing the increments to decrease till they vanish, the limit readily appears.

503. For example, let a be an invariable quantity, x a flowing quantity, and o any increment of x; then the simultaneous increments of xx and ax will be $2xo+oo$ and ao, which are in the same ratio to each other as $2x+o$ is to a. This ratio of $2x+o$ to a continually decreases while o decreases, and is always greater than the ratio of $2x$ to a while o is any real increment, but it is manifest that it continually approaches to the ratio of $2x$ to a as its limit; whence it follows that the fluxion of xx is to the fluxion of ax as $2x$ is to a. If x be supposed to flow uniformly, ax will likewise flow uniformly, but xx with a motion continually accelerated: the motion with which ax flows may be measured by ao, but the motion with which xx flows is not to be measured by its increment $2xo+oo$ (by ax. 1), but by the part $2xo$ only, which is generated in consequence of that motion; and the part oo is to be rejected because it is generated in consequence only of the acceleration of the motion with which the variable square flows, while o the increment of its side is generated: and the ratio of $2xo$ to ao is that of $2x$ to a, which was found to be the limit of the ratio of the increments $2xo+oo$ and ao (*fig.* 220). In general, if (as in art. 66, &c.) the point P be supposed to describe DG upon the right line Aa with an uniform motion, and M describe LS upon Ee with a variable motion in the same time, the velocity of M at L be to the constant velocity of P as Lc is to Dg, and Lf be always to Dg as LS to DG; it was shown in those articles that if LS and DG (the simultaneous increments of EL and AD) be supposed to decrease till they vanish, then the ratio of Lf (or L$c \mp cf$) to Dg, or of LS to DG, will approach continually to that of Lc to Dg as its limit. Therefore if the ratio be determined, which is the limit of the various proportions in which Lf is to Dg while the increments LS and DG decrease till they vanish, this can be no other than the ratio of Lc to Dg, or of the velocity of M at the term when it comes to L to the constant velocity of P, that is of the fluxion of EL to the fluxion of AD. If LC be to CS as Lc is to Cf, then LC will be the part of LC $\mp$ CS (the expression of LS) which arises in consequence of the motion

motion of M at L, and CS the part which arises in consequence of the variation of the motion of M while it describes LS.

504. This limit is discovered by any method that serves to distinguish the two parts Lc and *cf* of $\overline{Lc + cf}$ the expression of L*f*, or LC and CS the two parts of $\overline{LC + CS}$ the expression of LS, from each other; of which parts the former measures the motion of M at L, while the latter arises from the variation of the motion of M while it describes LS. We distinguished these parts from each other by this property, in the preceding chapters. But since it is the property of the part *cf* to decrease, and at length to vanish, with the increments LS and DG, while Lc remains, it appears to be a just as well as concise method of investigating this limit, to suppose the increments to decrease, to find what part of the expression of L*f* decreases, and at length vanishes with them, to reject this part, and retain the other Lc only for measuring the velocity of M at L. It is objected against Sir *Isaac Newton*'s method of investigating this limit, that he first supposes that there are increments (as LS and DG), that when it is said *let the increments vanish,* the former supposition is destroyed, and yet a consequence of this supposition, *i. e.* an expression got by virtue thereof, is retained. But the suppositions that are made in this method of investigating the limit are not so contradictory as this objection seems to import. He first supposes that there are increments generated, and represents their ratio by that of two quantities (as L*f* and D*g*), one of which (D*g*) is given so as not to vary with the increments. If he had afterwards supposed that no increments had been generated, this indeed had been a supposition directly contradictory to the former. But when he supposes those increments to be diminished till they vanish, this supposition surely cannot be said to be so contradictory to the former, as to hinder us from knowing what was the ratio of those increments at any term of the time while they had a real existence, how this ratio varied, and to what limit it approached, while the increments were continually diminished. On the contrary, this is a very concise and just method of discovering the limit which is required. It had been easy, if it had been of any use, to have supposed the generating motions to have proceeded in their course; and to have substitut-

ed,

ed, in place of his decreasing increments, quantities that should decrease so as to be always in the same ratio to each other as the increments were while they were generated, But this was not necessary, and it is to be remembered that the ratio Lc to Dg, the limit of the variable ratio of Lf to Dg, is not proposed as the ratio of increments that have vanished, but as the ratio of the velocities with which the points M and P did set out from L and D to generate real increments.

505. The ratio of Lc to Dg is likewise called the *first* or *prime ratio* of the increments LS and DG; because though the ratio of those increments continually varies when the motion of M is continually accelerated or retarded, yet the ratio of the generating motions (or that of Lc to Dg) is the term or limit from which the variable ratio of the increments proceeds, or sets out, to increase or decrease. This ratio, strictly speaking, is not the ratio of any real increments whatsoever, because any increment LS partly depends on the motion of M at L, and partly on the continual acceleration or retardation of its motion from that term. But as the tangent of an arch is the right line that limits the position of all the secants that can pass through the point of contact (art. 181), though, strictly speaking, it be no secant, so a ratio may limit the variable ratios of the increments, though it cannot be said to be the ratio of any real increments. The ratio of the generating motions may be likewise said to be the *last* or *ultimate ratio* of the increments while they are supposed to be diminished till they vanish, for a like reason.

506. Most of the propositions in the preceding chapters may be briefly demonstrated by this method. It will be sufficient to give a few examples (*fig.* 38). First, let us resume the construction in art. 140, where SA is invariable, SA, AP and AL are in continued proportion, and it is required to find the ratio of the fluxion of AL to the fluxion of AP. Because Ll the increment of AL is to Pp the increment of AP as DL is to SP, and the angle PSD is always equal to pSA, it is manifest that if those increments Ll and Pp be supposed to be diminished till they vanish, the angle PSD will approach to PSA, and at length coincide with it, PD will become equal to PL and DL to 2PL; so that the

the ultimate ratio of Ll to Pp must be that of 2PL to SP, or of 2AP to SA; and the fluxion of AL must be to the fluxion of AP in the same ratio. In the same manner SA, AP, AL, and AM being in continued proportion, Mm the increment of AM is to Pp as GM to SP; and when these increments are diminished till they vanish, GL becomes equal to 2LM, and GM to 3LM; so that the last ratio of Mm to Pp is that of 3LM to SP, or that of 3AL to SA; and the fluxion of AM is to the fluxion of AP in the same ratio. In like manner the 8th and 9th propositions may be deduced.

507. In prop. 14, where AD(*fig.* 47) is the base, DE the ordinate, DG the increment of the base, IH the simultaneous increment of the ordinate, if DG be supposed to be diminished till it vanish, the angle HET (contained by the chord EH and tangent ET) decreases till it vanish, by art. 181; and the ultimate ratio of DG to IH is that of EI to IT, which is therefore the ratio of the fluxion of the base AD to the fluxion of the ordinate. The ultimate ratio of the arch EH to the tangent ET is a ratio of equality, and the fluxion of the curve is to the fluxion of the base as ET to EI. In the same manner the 15th, 16th, and 17th propositions may be briefly deduced.

508. In prop. 18, a circle described through C, E, and K(*fig.* 61, and 62) touches the right line AE, because the angle ECK is made equal to SEA. Therefore when P approaches to E till it coincide with it, the ultimate ratio of the angle PKE to PCE is a ratio of equality, and the ultimate ratio of the angle PCE to the angle PSE is that of SE to KE, or of ST to CT; whence the fluxion of the angle ACP is to the fluxion of aSP as ST is to CT.

509. If the point C(*fig.* 223) be taken upon the right line AB, that joins the centres of the bodies A and B, so that CA be to CB as the body B is to A, then C is the centre of gravity of A and B; if the point G be taken upon CE, so that GE be to GA as the sum of A and B is to the body E, then is G the centre of gravity of the three bodies, A, B, and E; and in the same manner the centre of gravity of any number of bodies is determined. Let kn be any right line, Aa, Bb, and Cc any parallel lines from A, B, and C that meet kn in a, b, and c; then the sum of the rectangles

angles contained by A and Aa, and by B and Bb, shall be equal to the rectangle contained by $A+B$ and Cc when A and B are on the same side of kn; but to the rectangle contained by $A-B$ and Cc when they are on different sides of kn; because if AV and Bv parallel to kn meet Cc in V and v, CV will be to Cv as CA to CB, or as B to A; and the rectangle $A\times CV$ equal to $B\times Cv$. It follows that if G be the centre of gravity of any number of bodies, the rectangle contained by Gg (any right line from G that meets a given plane kn in g) and the sum of all the bodies is equal to the aggregate of the rectangles contained by each body, and the parallel from it terminated always by kn, that is to the aggregate of $A\times Aa$, $B\times Bb$, $E\times Ee$, *&c.* in collecting which any rectangle is to be considered as negative, or to be subducted, when the body is not on the same side of kn with G (*fig.* 80). Hence, cor. 6, prop. 19, may be deduced (that the surface described by any line FNf revolving about the axis kn is equal to the rectangle contained by FNf and the line described by its centre of gravity C in the same time) by applying what has been shown of the bodies A, B, E, *&c.* to the elements of the arch FNf, and substituting this arch itself for the sum of the bodies. In the same manner it is shown that if G (*fig.* 225) be the centre of gravity of any figure DBd, kn a right line in the plane of this figure parallel to Dd and given in position, GA perpendicular to kn in A meet Mm any ordinate of this figure parallel to kn in P, then the solid contained by the area DBd and the perpendicular GA will be equal to the fluent of the solid contained by the rectangle which measures the fluxion of the area MBm and the perpendicular PA, by substituting the elements of the area for the bodies A, B, E, *&c.* and the whole area DBd for the sum of the bodies. And if G be the centre of gravity of a solid DBd, of which Mm represents any section parallel to Dd, let the whole solid be represented by S, the fluxion of the solid MBm by $\dot{s}$, and $GA\times S$ will be equal to the fluent of $PA\times\dot{s}$.

510. There are several theorems concerning the centre of gravity, and its motion, that are useful in the resolution of problems of various kinds, which we shall take this occasion to describe briefly. In any system of bodies the sum of their motions

tions when estimated in a given direction is equal to the motion of a body that is equal to the sum of those bodies, and proceeds with the velocity of their common centre of gravity, if its motion be reduced to the same direction (*fig.* 224). A motion that is as CL, in the direction CL, reduced to any other direction C*c* is measured by CP, if LP be perpendicular to C*c* in P. The same motion reduced to the opposite direction *c*C is still measured by CP, but is then considered as negative. Let the bodies A and B with their centre of gravity move in the same time into F, H, and L respectively; let F*f*, H*h*, and L*l* parallel to C*c* meet *kn* in *f*, *h*, and *l*; and FM, HN, LP parallel to *kn* meet A*a*, B*b*, C*c* in M, N, and P respectively; then since the sum of $A \times Aa + B \times Bb$ is equal to $\overline{A+B} \times Cc$, and $A \times Ff + B \times Hh$ is equal to $\overline{A+B} \times Ll$, it follows that $A \times AM + B \times BN$ is equal to $\overline{A+B} \times CP$. In the same manner $A \times FM + B \times HN$ is equal to $\overline{A+B} \times LP$. And in the same manner it appears that the aggregate of the motions of any number of bodies A, B, E, *&c.* is equal to the motion of their sum A+B+E, *&c.* proceeding with the velocity of their common centre of gravity, when these motions are all reduced to any one direction. It follows likewise that if the motions of the bodies are all uniform and rectilineal, the centre of gravity is either quiescent, or its motion is uniform and rectilineal. For in this case the ratio of the right lines AM, FM, BN, HN to each other being invariable, as well as the ratio of A to B, the ratio of CP to LP must be invariable.

511. As the aggregate of the motions of any number of bodies reduced to any given direction is never affected by the composition or resolution of their motions, or by any actions of those bodies upon one another that are mutual and equal in contrary directions, or by any powers that act equally upon them with opposite directions; so the motion of the centre of gravity of any system of bodies is never affected by their collisions, or when they attract or repel each other equally. In the same manner as the motion of any one body continues the same till some external force or resistance effect it, by Sir *Isaac Newton's* first law of motion; so the motion of the centre of gravity of any system of bodies continues the same unless some foreign

...507.

Fig. 38. Art. 506.

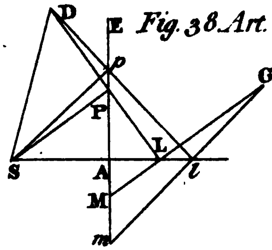


Fig. 80.

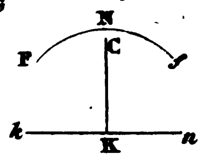


Fig. 149. Art. 553.

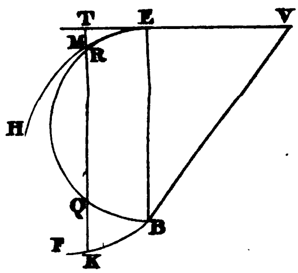
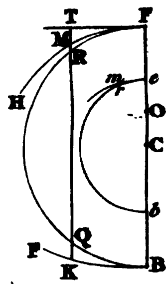


Fig. 152.



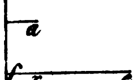
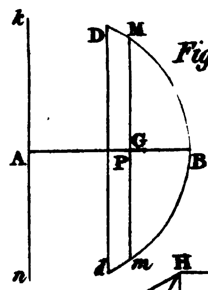


Fig. 225.



...g. 223.

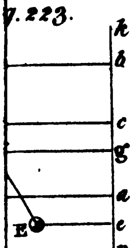
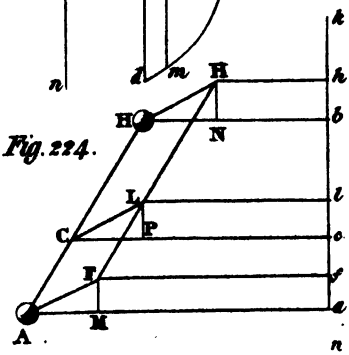


Fig. 224.



foreign influence disturb it. If there was any action without an equal and contrary reaction, the state of the centre of gravity of the system would be affected by it. And the equality of these being constantly confirmed by experience, it is not without ground that it is held to be a general law, and extended by Sir *Isaac Newton* to the gravitation of bodies. It is manifest however that it is not the sum of the absolute forces of bodies, without regard to the directions of their motions, that is preserved the same unalterable by their collisions, in consequence of the equality of action and reaction* (according to Sir *Isaac Newton's* third law of motion); since this is a general

* When it is said that *the quantity of absolute force is unalterable by the collision of bodies*, and *that this follows so evidently from the equality of action and reaction, that to endeavour to demonstrate it would only render it more obscure*, something else must be meant by action and reaction than has been generally understood by these terms, and that has not been explained by those philosophers. According to this doctrine it would seem that the equality of action and reaction should take place in the collisions of such bodies only as are perfectly elastic (that is of no bodies known in nature), and not even of these, unless we measure their forces by the compound ratio of the squares of their velocities and of their quantities of matter. And though this mensuration of the forces of bodies was admitted, the quantity of absolute force will be found to be so far affected by the collisions of bodies, that it must be less during the small time in which they act upon each other than before and after the stroke; whereas the quantity of motion estimated in the same direction is preserved the same while the bodies act upon each other as before and after, and never can suffer any change from their mutual actions. But as it might seem to be an improper digression if we should insist on this subject here, we shall only subjoin an illustration of an argument which was offered some time ago, to show, that we cannot abandon the old doctrine concerning the measures of the forces of bodies in motion, without exchanging plain principles that have been generally received concerning the actions of bodies upon the most simple and uncontested experiments, for notions that seem at best to be very obscure.

Let A and B (*fig.* 226) be two equal bodies that are separated from each other by springs interposed between them (or in any equivalent manner) in a space EFGH, which in the mean time proceeds uniformly in the direction BA (in which the springs act) with a velocity as 1; and suppose that the springs imprint on the equal bodies A and B equal velocities in opposite directions that are each as 1. Then the absolute velocity of A (which was as 1) will be now as 2; and, according to the new doctrine, its force as 4: whereas the absolute velocity and the force of B (which was as 1) will be now destroyed; so that the action of the springs adds to A a force as 3, and subducts from the equal body B a force as 1 only; and yet it seems manifest that the actions of the springs on these equal bodies

neral law, and extends to hard and soft bodies as well as to such as are perfectly elastic, and the sum of the absolute motions of those cannot be said to be unalterable by their collisions. It is the quantity of motion estimated in the same direction that is preserved the same without any change from any mutual actions of bodies in consequence of the equality of action and reaction. But we proceed to give some instances of the use of those theorems in the resolution of problems.

512. From these principles the effects of the collisions of bodies are readily determined. The bodies A and B (*fig.* 227) being supposed void of elasticity, let C be their centre of gravity, and their velocities before the stroke be represented by AD and BD respectively. Then supposing the stroke to be direct, they will proceed together after it as one mass, and consequently with the velocity CD of their centre of gravity. But if the bodies are perfectly elastic, take CE equal to CD in an opposite direction, and the velocities of A and B after the stroke will be represented by EA and EB respectively, the change produced

bodies ought to be equal. In general, if m represent the velocity of the space EFGH in the direction BA, n the velocity added to that of A and subducted from that of B by the action of the springs, then the absolute velocities of A and B will be represented by $m+n$ and $m-n$ respectively, the force added to A by the springs will be $2mn+nn$, and the force taken from B will be $2mn-nn$, which differ by $2nn$. Further it is allowed that the actions of bodies upon one another are the same in a space that proceeds with an uniform motion as if the space was at rest (*la force du choc, ou l'action des corps les unis sur les autres, depend uniquement de leur vitesses respectives. Discours sur le mouvement, Paris,* 1726). But if the space EFGH was at rest, the forces communicated by the springs to A and B had been equal, and the force of each had been represented by nn. These arguments are simple and obvious, and seem on that account to be the more proper in treating of this question. Though there are certain effects produced by the forces of bodies that are in the duplicate ratio of their velocities, we are not thence to conclude that the forces themselves are in that ratio, no more than we are to conclude that a force which would carry a body upwards 500 miles in a minute is infinite, because it may be demonstrated (if we abstract from the resistance of the air) that a body projected with this velocity would rise for ever, and never return to the earth. And as reaction is only equal to action when both are estimated in opposite directions upon the same right line, so we are never to estimate the force which one body loses or acquires by that which is produced or destroyed in another body in a different direction; whence the objections against the usual manner of measuring the forces of bodies may be resolved, and even improved for to support it.

in

in their velocities in this case by the stroke being double of what it was in the former, the difference of AD and CD being equal to the difference of CD (or CE) and EA, and the difference of CD and BD equal to the difference of EB and CD. If B have no motion before the stroke, then CE is to be taken equal to CB, the velocity of A before the stroke being represented by AB. In this case if the right line oa be to ob as A is to B, and ab be bisected in e, the velocity of A before the stroke will be to that of B after the stroke as half the sum of A and B is to A, or as oe is to oa. And if motion be communicated in this manner from the body A to a series of bodies in geometrical progression, of which A and B are the first terms, then the velocities successively communicated to those bodies will be in a geometrical progression, the common ratio of any two subsequent terms will be that of oe to oa; and, if n be the number of bodies ~~without including the first~~ A, the velocity of the last will be to the velocity of the first as the power of oa whose exponent is n is to the same power of oe. Therefore if od represent the last body in the progression, and ov the velocity communicated to it, the velocity of the first oa being represented by oa, and oa be the *modulus* of the system, the logarithm of od will be to that of ov as the logarithm of ob is to that of oe, because the logarithm of od is to that of ob as n is to 1, and the logarithm of ov is to that of oe in the same ratio.

513. Any three bodies being represented by oa, ob, and od, let the first strike the second supposed at rest before the stroke, and the second strike the third quiescent, let of be to od as oa is to ob; and the velocity communicated in this manner to the third shall be to the velocity of the first as oa is to one fourth part of the sum of oa, ob, of, and od. For the velocity of the first oa is to the velocity of the second ob as the sum of oa and ob to $2oa$; the velocity of ob is to that of od as the sum of ob and od to $2ob$; consequently the velocity of the first oa is to the velocity of the third od in the compound ratio of $oa+ob$ to $2oa$ and of $ob+od$ to $2ob$, that is (since oa, ob, of, od, are proportional, so that oa is to ob as $oa+of$ to $ob+od$, and $oa+ob$ to ob as the sum of oa, ob, of, and od to $ob+od$) as the sum of oa, ob, of, and od is to $4oa$. Hence the velocity of oa

being given, the velocity communicated to *od* is inversely as the sum of *oa*, *ob*, *of*, and *od*, and is greatest when this sum is least, that is, if *oa* and *od* be given, when *ob* and *of* coincide with each other and with the mean proportional betwixt *oa* and *od*. Therefore the velocity communicated to *od* is greatest when *ob* the body interposed betwixt *oa* and *od* is a mean proportional between them. This is one of Mr. *Huygens*'s theorems, from which it follows, that the more such geometrical mean proportionals are interposed betwixt *oa* and *od*, the greater is the velocity communicated to *od*.

514. There is however a limit which the velocity communicated to *od* never amounts to (the bodies *oa*, *od*, and the velocity of *oa* before the stroke being given), to which it approaches continually while the number of such bodies interposed between *oa* and *od* is always increased. And this limit is a velocity which is to the velocity of the first body *oa* before the stroke in the subduplicate ratio of *oa* to *od*. This limit is not mentioned by Mr. *Huygens*, but may be determined from art. 512 and 179. For while *ab* is continually diminished, and *ob* approaches to *oa*, the last ratio of the logarithm of *ob* to *ab*, or of the logarithm of *oe* to *ae*, is a ratio of equality, by art. 179; consequently the logarithm of *ob* becomes ultimately double of that of *oe*, and (by art. 512) the logarithm of *od* double of that of *ov*. Therefore if *ok* be a mean proportion betwixt *oa* and *od*, the logarithm of *ov* will become equal to the logarithm of *ok*, but with a contrary sign; so that *ok*, *oa*, and *ov* will be in continued proportion: and *ov* the velocity of the last body *od* will be to *oa* the velocity of the first *oa* as *oa* is to *ok*, or in the subduplicate ratio of the first body *oa* to the last *od*.

515. The same principles will serve for determining the effects of the collision, when a body strikes any number of bodies at once in any directions whatever. Let the bodies first be perfectly hard and void of elasticity, and the body C (*fig.* 228) moving in the direction CD with a velocity represented by CD strike at once the bodies A, B, E, *&c.* that are supposed at rest before the stroke in directions CF, CH, CK, *&c.* in the same plane with CD, and D*a*, D*b*, D*e*, be perpendicular to CF, CH, CK in *a*, *b*, and *e* respectively. Determine the point P

where

where the common centre of gravity of the bodies C, A, B, E, &c. would be found if their centres were placed at the points C, *a*, *b*, *e*, &c. respectively, by art. 509, join DP, and CL parallel to DP shall be the direction of the body C after the stroke. Let PR perpendicular to DP meet CD in R, and DL perpendicular to CD meet CL in L; then if CL be divided in G so that CG be to GL in the ratio compounded of that of CD to CR and that of the body C to the sum of all the bodies, the velocity of C after the stroke will be represented by CG; that is, the velocity of C after the stroke will be to its velocity before it as CG is to CD. Let G*f*, G*h*, and G*k*, be respectively perpendicular to CF, CH, and CK in *f*, *h*, and *k*, and the velocities of A, B, and E, after the stroke will be represented by C*f*, C*h*, and C*k*. But if we now suppose the bodies to be perfectly elastic, or the relative velocities before and after the stroke be always equal when ~~measured on~~ the same right line, produce DG till D*g* be equal to 2DG, join C*g*, and the body C will describe C*g* after the stroke in the same time that it would have described a right line equal to CD before the stroke. And in like manner the motions are determined when the elasticity is imperfect, if the relative velocity after the stroke is always in a given ratio to the relative velocity before it in the same right line. Mr. *Bernouilli* has deduced the computations of the motions in the case when the bodies are perfectly elastic, and there are bodies on one side of the line of direction CD that are always respectively equal to those on the other side, and are impelled in directions that form equal angles with CD in the same plane, from the principle that the sum of the bodies multiplied by the squares of the velocities is the same before and after the stroke; which computations will be found to agree with what we have shown, by supposing DP and CL to fall upon CD, and restricting our supposition in other respects so as it may agree to this case. These problems being represented as of an uncommon difficulty, it may be worth while to subjoin the following construction which is still more general, and is deduced from the principles in art. 510 and 511.

516. Let the bodies C, A, B, E, &c. (*fig*. 229) move now in the directions CD, CF, CH, CK, &c. in one plane with

 velocities

velocities represented by CD, Ca, Cb, Ce, *&c.* and the body C overtake and strike them at once in these directions. Let T be the point where the common centre of gravity of all the bodies C, A, B, E, *&c.* would be found if they were placed in D, a, b, e, *&c.* respectively; let T*a*, T*b*, T*e*, *&c.* be perpendicular to CF, CH, CK, *&c.* in *a, b, e, &c.* and P be the point where their common centre of gravity would be found if the bodies were placed at C, *a, b, e, &c.* respectively; join TP, and CL parallel to TP will be the direction of C after the stroke when all the bodies are supposed perfectly hard and void of elasticity. Let PR perpendicular to TP meet CT in R, and TL perpendicular to CT meet CL in L; let CS be to CT as the body C is to the sum of all the bodies; upon CL take CG in the same ratio to CL as CT is to CS+CR, and CG will represent the velocity of C after the stroke; whence the velocities of the other bodies in their respective directions CF, CH, CK, *&c.* are determined as before. We omit some other theorems of this kind where the directions are in different planes, because they would lead us too far from our principal subject. When the bodies are perfectly elastic, join DG, and upon it take D*g*, double of DG; but if the elasticity be imperfect, and the respective velocity after the stroke be in a given ratio to the respective velocity before the stroke, upon DG produced take G*g* to DG in that given ratio; and C*g* will represent the direction and velocity of C after the stroke; whence it is easy to determine the velocities of the other bodies. The other cases of this problem are resolved in like manner from the same principles.

517. Mr. *Huygens* has shown that in the collisions of two bodies which are perfectly elastic, the sum of the bodies multiplied by the squares of their velocities is the same after the stroke as before it. It is justly observed that this proposition is so far general as to obtain in all collisions of bodies that are perfectly elastic; but as this cannot be held an immediate consequence of the equality of action and reaction, as was observed above, and it is by some considered as a theorem of great use, we shall show how it may be demonstrated when a body strikes any number of bodies at once, as in art. 515. Let DQ, *gq*,

fm,

fm, hn, kr be perpendicular to CG in Q, q, m, n and r (*fig.* 228). Then the rectangles contained by Cm and CG, Cn and CG, Cr and CG will be respectively equal to the squares of Cf, Ch, and Ck. If the bodies C, A, B, E be supposed to have no elasticity, their velocities after the stroke will be represented by CG, Cf, Ch, and Ck, the velocity of C before the stroke being represented by CD; because in this case no relative velocity is generated by the stroke in their respective directions; and the sum of $A \times Cm$, $B \times Cn$, $E \times Cr$ is equal to $C \times GQ$, because the sum of the motions which would be communicated to A, B, and E in the direction CG is equal to the motion which C would lose in the same direction by art. 511. Therefore the sum of $A \times Cf^2$, $B \times Ch^2$, $E \times Ck^2$ is equal to $C \times CG \times GQ$; and to these if we add $C \times CG^2$, the sum of all the bodies multiplied by the squares of their velocities in this case would be $C \times CG \times CQ$. But when the bodies are supposed to be perfectly elastic, the velocities of A, B, and E are to be represented by $2Cf$, $2Ch$, and $2Ck$ respectively; the sum of $A \times 4Cf^2$, $B \times 4Ch^2$ and $E \times 4Ck^2$ is equal to $C + 4CG + GQ$ or (*elem.* 8, 2) $C + CQ^2 - C \times Cq^2$; to which if we add $C \times Cg^2$ (or $C \times Cq^2 + C \times GQ^2$) the whole sum of the products when each body is multiplied by the square of its velocity is equal to $C \times CD^2$; and consequently is the same after the stroke as it was before the stroke. But when the bodies are void of elasticity, this sum is less after the stroke than before it in the ratio of $CG + CQ$ to CD^2 or of CG to CL. The same proposition is demonstrated in like manner of perfectly elastic bodies in the case of art. 516. And when the bodies A, B, E move before the stroke in directions different from those in which C acts upon them, the proposition will appear by resolving their motions into such as are in those directions (which alone are affected by the stroke), and such as are in perpendiculars to those directions, from *elem.* 47, 1. This proposition likewise holds when bodies of a perfect elasticity strike any immoveable obstacle as well as when they strike one another, or when they are constrained by any power or resistance to move in directions different from those in which they impel one another, as we shall show afterwards. But it is manifest that it is not to be held a general principle or law of

motion, since it can take place in the collisions of one sort of bodies only. The solutions of some problems which have been deduced from it may be obtained in a general and direct manner from plain principles that are universally allowed, by determining first the motions of hard bodies which are supposed to have no elasticity, and thence deducing the solutions of other cases when the relative velocities before and after the stroke are equal, or in any given ratio. It will be said perhaps that there are no such bodies known in nature. But though no bodies that are perfectly elastic, or no mathematical fluid be known in nature, to investigate their motions is allowed to be an useful inquiry. It is a consequence however from the proposition we have described, that while perfectly elastic bodies move in any manner, if any new force act upon them that generates equal velocities in the same direction in each, the excess of the sum of the products of each body multiplied by the square of its velocity, above the product of the sum of the bodies multiplied by the square of the velocity of their common centre of gravity, is not affected by this new force or by their collisions.

518. Suppose now that the body C (*fig.* 230) moving in the direction CD with the velocity CD impels the bodies A and B in the directions CF and CH; but that A and B cannot move in those directions, being constrained to move in the respective directions C*f* and C*h*, by planes parallel to C*f* and C*h* along which we suppose them to slide without friction, or by their being fixed to the extremities of lines OA and UB perpendicular to C*f* and C*h*A, and moveable about the centres O and U, or in any other equivalent manner. Suppose all those lines to be in the same plane with CD, and that A and B were at rest before the stroke. Let D*a* and D*b* perpendicular to CF and CH meet C*f* and C*h* in *a* and *b* respectively; draw *a*F perpendicular to C*a*, and *b*H perpendicular to C*b*, meeting CF and CH in F and H, and F*m*, H*n* parallel to CD meeting D*a* and D*b* in *m* and *n*. Let P be the common centre of gravity of the bodies C, A, and B when their respective centres are supposed to be placed at C, *m*, and *n*, join DP, and CL parallel to DP shall be the direction of C after the stroke, the bodies being supposed to be perfectly hard and void of elasticity. Let *p* be the common centre of

gravity

gravity of C, A, and B when their respective centres are supposed to be placed at D, F, and H; draw *pr* perpendicular to DP meeting CD in *r*, let CS be to CD as the body C is to the sum of all the bodies; let DL perpendicular to CD meet CL in L, join *r*L, and let SG parallel to *r*L meet CL in G, then CG will represent the velocity of C after the stroke; and if G*f* and G*h* respectively perpendicular to CF and CH meet C*f* and C*h* in *f* and *h*, then C*f* and C*h* will represent the velocities of A and B after the stroke.

519. When the bodies are perfectly elastic, C*g* the direction and velocity of C is found as in art. 515, by producing DG till D*g* be equal to 2DG. In this case, though the motion of the centre of gravity, or the sum of the motions of the bodies in the direction CD, be diminished by the stroke (because of the resistance of the planes or lines by which the bodies A and B are hindered to move in the directions CF and CH in which C impels them, and constrained to move in the directions C*f* and C*h*), yet the sum of the products of the bodies multiplied by the squares of their velocities is the same after the stroke as before it. For let *hf* perpendicular to C*h* meet CH in *f*, and *fu* perpendicular to C*f* meet CF in *u*; draw *fx*, *uz*, DQ, and *gq* perpendicular to CL in *x*, *z*, Q, and *q*; join *zf*, and the angle C*fz* being equal to C*uz* or CG*f*, the triangles C*zf* and C*f*G are similar, and the rectangle GC*z* equal to the square of C*f*. In the same manner the rectangle GC*x* is equal to the square of C*h*; therefore the sum of $A \times 4Cf^2$ and $B \times 4Ch^2$ is equal to the product of $A \times Cz + B \times Cx$ by $4CG$. But $A \times Cz + B \times Cx$ is the quantity of motion which C loses in the direction CL when it communicates to A and B velocities C*f* and C*h* in their respective directions C*f* and C*h*, by impelling them in the directions CF and CH, and therefore is equal to $C \times GQ$ by art. 511. Therefore since $C \times GQ \times 4CG$ is equal to $C \times CQ^2 - C \times Cq^2$, if we add $C \times Cg^2$, the whole sum of the products of the bodies multiplied by the squares of their velocities after the stroke will be $C \times CD^2$, the same as it was before the stroke; and it is manifest that this demonstration is applicable when C strikes any number of bodies in any directions whatsoever.

520. The demonstration of the constructions in art. 515, 516, and 518 will easily appear if we subjoin that of the first in art. 515 (*fig.* 231). Resuming therefore the construction in that article, suppose moreover that Lp, Lq and Lt are perpendicular to CF, CH and CK in p, q, and t; draw aM, pm, bN, qn, eZ, tz, and PV perpendicular to CD in M, m, N, n, Z, z, and V. Let the sum of the bodies C, A, B, and E be expressed by S, and since P was the centre of gravity of the bodies C, A, B, and E when their respective centres were supposed to be placed at C, a, b and e, it follows from art. 509 that $S \times PV$ will be equal to $A \times aM + E \times eZ - B \times bN$, and $S \times DV$ equal to $C \times CD + A \times DM + B \times DN + E \times DZ$. If we suppose the bodies to be void of elasticity, or no relative velocity to be generated by the collision in their respective directions, then while C describes CL the bodies A, B, and E will describe right lines respectively equal to Cp, Cq, and Ct. Therefore if we suppose CL and CH to be on one side of CD, and CF and CK to be on the other side of it, $C \times DL + B \times qn$ will be equal to $A \times pm + E \times tz$, by art. 510, because the centre of gravity of the bodies had no motion in the direction perpendicular to CD before the stroke, and consequently has no motion in that direction after it. Let Lp meet aM in r, and pu parallel to CD meet aM in u, then au will be to ar (or DL) as DM to CD, and $au \times CD$ equal to $DL \times DM$, so that $A \times CD \times pm$ will be equal to $A \times CD \times aM - A \times DM \times DL$. In the same manner $B \times CD \times qn$ will be equal to $B \times CD \times bN + B \times DN \times DL$, and $E \times CD \times tz$ equal to $E \times CD \times eZ - E \times DZ \times DL$. From which it follows that $C \times CD \times DL + B \times CD \times bN + B \times DN \times DL$ is equal to $A \times CD \times aM - A \times DM \times DL + E \times CD \times eZ - E \times DZ \times DL$, or $S \times DV \times DL$ equal to $S \times CD \times PV$. Therefore CD is to DL as DV to PV, and CL is parallel to DP. The direction of C after the stroke being thus determined, suppose that CG is to CD as the velocity of C after the stroke to its velocity before the stroke. Then because the sum of the motions of the bodies estimated in any given direction is not affected by the stroke, or the motion of their common centre of gravity is uniform, this

sum

sum will be equal to $C \times CD$ in the time C describes CG; and $C \times CD + A \times Cm + B \times Cn + E \times Cz$ will be to $C \times CD$ as the time in which C describes CL to the time in which it describes CG, or as CL to CG. Therefore since $CD \times Mm$ is equal to $aM \times DL$, $CD \times Nn$ to $bN \times DL$, and $CD \times Zz$ to $eZ \times DL$, it follows that $A \times CD \times CM - A \times aM \times DL + B \times CD \times CN + B \times bN \times DL + E \times CD \times CZ - E \times eZ \times DL$ is to $C \times CD^2$ as LG is to CG; that is, $S \times CD \times CV - S \times PV \times DL$ is to CD^2 as LG is to CG. But PR is perpendicular to DP by the construction, and CD to DL as PV to VR, or $PV \times DL$ equal to $CD \times VR$; consequently $S \times CR$ is to $C \times CD$ as LG it to CG. Therefore CG the velocity of C after the stroke is determined by dividing CL in G, so that CG may be to LG in the compound ratio of CD to CR, and of C to S the sum of all the bodies; which was the solution given in art. *515, when ~~the bodies were~~ supposed* to have no elasticity, so that no relative velocity of C and the other bodies was generated in their respective directions.* In the same manner it is shown in the case described in art. 516, that CS being to CT as C is to

* Because the points *a*, *b*, *e*, &c. (*fig.* 231, *N.* 2) are always in the circumference of a circle described upon the diameter CD, if we should suppose a sphere C to strike equal homogeneous particles that touch it in an ark AB which is in the same plane with CD, the sum of those particles be called Q, CA and CB meet the circle *CaD* in *a* and *b*, the point X be the centre of gravity of the ark *aDb*, and CX be divided in P so that CP be to CX as Q is to C + Q, the direction of C after the stroke will be parallel to DP. If AB be a semicircle bisected by CD, the point X will be the centre of the circle *CaD*; and the velocity which C loses by the stroke will be to its incident velocity as Q to 2C+Q. But because the resistance of a sphere in a fluid is not discovered in this manner (*Newt. Princip. lib.* 2. *prop.* 32, &c. *schol.*), we have not insisted on those cases.

These theorems are given from a treatise concerning the mensuration of the force of bodies in motion and the effects of their collisions, written in 1728 (by way of supplement to a small piece printed on this subject at *Paris* 1724) that was then communicated to several persons, and intended to have been published; wherein I endeavour to show, that according to those who measure the forces of bodies by the squares of their velocities, equal actions generate unequal forces in equal times, and equal forces in unequal times, and that the force of a body must be said to have no greater effect in the direction of its motion than in other directions, and that several other suppositions must be admitted contrary to what has been generally agreed on. But after considering that these would perhaps be allowed with explications by such as favour that opinion, and

to S, CG is to CL as CT to CS + CR (*fig.* 229). In art. 518, the sum of the motions (*fig.* 230) estimated in the direction CD is not the same after the stroke as before the stroke; and while any body A acquires the velocity represented by C*f* in the direction C*f* from the impulse of C in the direction CF, we are to suppose that in generating this motion C loses a motion represented by A × C*u* in the direction CF, *fu* perpendicular to C*f* in *f* being supposed to meet CF in *u*. The motion A × *fu* is lost by the resistance of the plane or line that constrains the body A to move in the direction C*f* instead of CF in which it is impelled by the body C. By reducing the motions A × *fu* and B × *hf* to the direction CD, and adding them to the motions of C, A and B in that direction after the stroke, (or more briefly by reducing the motions A × C*u* and B × C*f* to

and that it is often proposed as a definition or axiom by them, that the force of a body in motion is measured by the number of springs which can produce or destroy it (though the same springs act for a longer time on a greater body than a lesser, and thereby generate in it a greater quantity of motion), I was unwilling to engage in a dispute that was perplexed by such suppositions, and that after all might seem to be in a great measure about words. And one of the chief designs of this chapter being to describe some general principles that are of use in the resolution of problems, this seemed to be a proper opportunity of publishing what was most material in that treatise. Therefore I have endeavoured to show in these and the following articles, that the consideration of the motions of hard bodies that have no elasticity (which are rejected for the sake of what is called the law of *continuity*, and is supposed to be general without sufficient ground), is of use in order to obtain general solutions; that the principle which Mr. *Huygens* calls the *conservatio vis ascendentis* (in his observations on some pieces concerning the centre of oscillation, *Oper.* vol. I. p. 248, Edit. *Lugd. Batav.* 1724), and which seems to be much the same with what is called the *conservatio vis vivæ* of late, obtains indeed in many cases besides those he has considered, and may be of use in several inquiries concerning the motions of bodies that have no elasticity, as well as those that are perfectly elastic, but is not general; and that there is no occasion to perplex the common doctrine concerning the action and reaction of bodies, or the mensuration of their force, for the sake of this principle when it takes place. They who hold this principle to be general confine this theory too much to one sort of bodies, which for any thing appears from nature have no prerogative above others. And while some insist on the preservation of the same quantity of absolute force in the universe with much warmth against Sir *Isaac Newton*, there is nevertheless no proposition in experimental philosophy more evident than that in many cases force is lost or diminished in the collisions of bodies from the weakness of their elasticity, whether we measure it by the velocities or by the squares of the velocities. And there is ground to think that it will not be generally allowed to be so easy a matter as they seem to imagine to give a satisfactory account how this can be reconciled with a principle so contradictory to it.

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the same direction) and supposing the sum equal to $C \times CD$, which was the sum of the motions in that direction before the stroke, the solution in art. 518, will appear. But we proceed now to show how these theorems may be applied for determining the motions of bodies that descend by their gravity, and at the same time impel other bodies, which will lead us to consider Mr. *Huygens*'s principle concerning what he calls their *vis ascendens*.

521. Let the accelerating force (*fig.* 228) and direction of gravity be always represented by CD, and let C by its gravity impel A, B and E (which we supposed at present to be avoid of gravity) in the respective directions CF, CH and CK from the beginning of its descent. If nothing hinder the bodies from giving way in those directions, and if these sides of A, B and E which C acts upon be planes perpendicular to CF, CH and CK (that C while it descends may impel them always in the same directions) then C will descend in the right line CL that was determined in art. 515; for CL the direction of C which was determined in that article does not depend upon the incident velocity of C, but only upon the quantities of matter in the several bodies, the direction of the motion of C, and those in which it acts upon the other bodies; and when these remain the direction of C after the stroke is always the same. The forces that accelerate the motions of C, A, B and E in their respectve directions CL, CF, CH and CK will be to the accelerating force of gravity, and the respective velocities that will be acquired by them to the velocity that would be acquired in an equal time by a body falling freely in the vertical line CD by its gravity, as CG, Cf, Ch, and Ck to CD. Let Gd be perpendicular to CD in d, and the sum of the products $C \times CG^2$, $A \times Cf^2$, $B \times Ch^2$, $E \times Ck^2$ will be equal to $C \times CD \times Cd$; for it was shown in art. 517, that $C \times CG^2 + A \times Cf^2 + B \times Ch^2 + E \times Ck^2$ is equal to $C \times CG \times CQ$, which (because CG is to Cd as CD to CQ) is equal to $C \times CD \times Cd$. But if the velocity which C acquires while it descends from C to G, and is accelerated by the force CG, be represented by CG, or its square by CG^2, the square of the velocity which it would acquire by falling freely in the vertical from C to d by its gravity CD will be

represented by $CD \times Cd$ (art. 434). Therefore the sum of the products which arise by multiplying each body by the square of the velocity which it acquires is equal to the product of the body C (which alone is supposed to gravitate) multiplied by the square of the velocity which it would acquire by falling freely from C to *d*, or by descending freely along the inclined plane CG. The same theorem holds when the directions vary in which the body C acts upon A, B and E while it descends; the demonstration of which will be comprehended in a more general case afterwards.

522. If the body C impel A and B (*fig.* 230) by its gravity in the directions CF and CH from the beginning of its descent, but these bodies be constrained, as in art. 518, to move in the directions C*f* and C*h*, the direction of the motion of C and the velocities that will be acquired by the respective bodies may be determined from what was shown in that article, the sides of A and B upon which C acts being planes, so that C may descend in the same right line CL and always impel A and B in the same directions CF and CH. The velocities acquired by C, A, and B at G, *f* and *h* will be to the velocity that would be acquired in an equal time by a body falling freely in the perpendicular as CG, C*f* and C*h* to CD. And because $C \times CG^2 + A \times Cf^2 + B \times Ch$ is equal to $C \times CG \times CQ$ (by what was shown in art. 519), or to $C \times CD \times Cd$, G*d* being perpendicular to CD in *d*; therefore the sum of the products of the bodies multiplied by the squares of the velocities which they acquire is in this case likewise equal to the product of C multiplied by the square of the velocity which it would acquire by the same descent C*d* if it fell freely in the vertical CD.

523 (*Fig.* 232, *N.* 1). To give some examples of this last case. If the body C impel by its gravity one body A only that is terminated by a plane perpendicular to CF, and A slide along a plane parallel to C*f* without friction, let D*a* perpendicular to CF meet C*f* in *a*, *a*F perpendicular to C*a* meet CF in F, and F*m* parallel to CD meet D*a* in *m*; upon C*m* take CP to C*m* as A is to C + A, join DP, and a right line from D parallel to CF will intersect CL parallel to DP in G. If in this case C*f* be supposed horizontal or perpendicular to CD, C*m* will coincide with C*a*, and CP

CP being taken upon C*a*, in the same ratio to C*a* as A is to C+A, a right line from D parallel to CF will intersect CL parallel to DP in G, so that CG will represent the direction in which C will descend and the force that accelerates its motion; and CG will be described by it in the same time that it would have described CD by falling freely in the vertical. If we suppose CF (*fig.* 232, *N.* 2.) to coincide with the vertical CD, D*a* will in this case be perpendicular to CD, and *a*F being perpendicular to C*a*, the point *m* will fall upon D, and CD is to be divided in G so that CG may be to DG in the compound ratio of C to A and of CD to CF, or of the square of CD to the square of C*a*. These last are the two cases considered by Mr. *Bernouilli* which have been lately published, *Comm. Acad. Petropol. tom.* 5; and these constructions agree with the computations which he deduces by resolving the force of C into two infinite progressions. If the body C impel in like manner two equal bodies A and B (*fig.* 232, *N.* 3) in directions CF and CH that form equal angles with the vertical, and *f*C*h* be one continued horizontal line, CD is to be divided in G, so that CG may be to GD in the compound ratio of C to the sum of the bodies A and B and of the duplicate ratio of the sine of the angle FCD to its cosine; and CG will represent the force that accelerates the motion of C, providing it always impel A and B in the same directions from the beginning of its descent.

524. The rest remaining as in art. 523, let us now suppose the bodies A and B (*fig.* 233) to gravitate as well as C. In this case the body C while it descends will have no effect upon the bodies A and B, unless the angles D*cf* and D*ch* exceed DCF and DCH respectively. The force and direction with which C descends being represented by CG, let G*f* and G*h* perpendicular to CF and CH (the respective directions in which C acts upon A and B) meet C*f* and C*h* in *f* and *h*, that C*f* and C*h* may represent the forces by which the motions of A and B are accelerated in the directions C*f* and C*h*. Let D*a* and D*b* perpendicular to C*f* and C*h* meet CF and CH in K and R respectively; let *fk* perpendicular to C*f* meet CF in *k*, and *hr* perpendicular to C*h* meet CH in *r*. Draw KM, *km*, RN, and *rn* perpendicular to CG in M, *m*, N, and *n*; and draw G*d*, *f*V, and *hv* perpendicular

dicular to the vertical line CD in d, V, and v. While C describes CG, A describes Cf; and because A would have described Ca in the same time by its own gravity, the part $A \times af$ of the force which produces the motion of A is what is generated in consequence of the action of C upon it; the force which C loses in the direction CF (in which it acts upon A) in generating this increase of the force of A in the direction Cf is $A \times Kk$, which reduced to the direction CG is $A \times Mm$. In the same manner the force which C loses in the direction CG by its action on B is $B \times Nn$. Let DQ be perpendicular to CG in Q, and the force with which C endeavours to descend in CG in consequence of its gravity being $C \times CQ$, it follows that $C \times CG + A \times Mm + B \times Nn$ is equal to $C \times CQ$, and $C \times CG^2 + A \times CG \times Mm + B \times CG \times Nn$ equal to $C \times CQ \times CG$ or $C \times CD \times Cd$. But the triangles Cmf, CfG being similar, Mm is to af as Cm to Cf or Cf to CG, and $Mm \times CG$ is equal to $Cf \times af$ or $Cf^2 - Cf \times Ca$, that is (because Cf is to CV as CD to Ca, and $Cf \times Ca$ is equal to $CD \times CV$) to $Cf^2 - CD \times CV$; and in the same manner $Nn \times CG$ is equal to $Ch^2 - CD \times Cv$. Therefore $C \times CG^2 + A \times Cf^2 - A \times CD \times CV + B \times Ch^2 - B \times CD \times Cv$ is equal to $C \times CD \times Cd$, or $C \times CG^2 + A \times Cf^2 + B \times Ch^2$ equal to $C \times CD \times Cd + A \times CD \times CV + B \times CD \times Cv$; that is (by art. 434), the sum of the products that arise by multiplying each body by the square of the velocity which it acquires is equal to the sum of the products when each body is multiplied by the square of the velocity which it would have acquired by the same perpendicular descent if it had fallen freely. But it must be observed, that if any body as A (for example) ascend, or the angle DCf be obtuse, then af is equal to $Cf + Ca$, and the term $A \times CD \times CV$ must be subducted in the latter part of the equation. The general theorem therefore is, that the sum of the products of the bodies multiplied by the squares of their respective velocities is equal to the difference of the products of those that descend multiplied by the squares of the respective velocities that would have been acquired by the same descents, and of the products of those that ascend multiplied by the squares of the respective velocities that would have been acquired by falling freely from the altitudes to which

which they have risen (*fig.* 233, *N.* 2). To give an example how the motions are determined in this case, suppose that C impels the equal bodies A and B in directions that form equal angles with the vertical CD, and that those bodies move in directions C*f* and C*h* that likewise form equal angles with the vertical greater than DCF or DCH. Let D*a* perpendicular to C*f* in *a* meet CF in K, *az* perpendicular to CF meet CD in *z*, and KN be perpendicular to CD in N; let D*e* perpendicular to CF meet C*f* in *e*; divide *ae* in *f*, so that *fe* may be to *af* in the compound ratio of CN to C*z*, and of the sum of A and B to C; then C*f* will represent the force that accelerates the motion of A or B; and *f*G perpendicular to CF will intersect CD in G, so that CG will represent the force which accelerates the motion of C. Let K*k* perpendicular to CF meet C*f* in *k*, and if C act upon one body A only, *ae* is to be divided in *f*, so that *fe* may be to *af* as A × C*k* to C × C*a*, and *f*G perpendicular to CF will intersect DG parallel to CF in G. But in these constructions we suppose that the body C acts upon the bodies A and B in invariable directions.

525. The same theorem takes place though the sides of A and B upon which C acts be not planes, and the directions vary in which they are impelled by the body C; providing C act upon them from the beginning of its descent, so that there be no collision or sudden communication of motion from one body to another. Let CG (*fig.* 234) the direction of C at any time meet C*f* the direction of A at the same time in C, and *ch* the direction of B in *c*; let CF be the direction in which C then acts upon A, and *c*H the direction in which it acts upon B; and the rest of the construction being similar to that in the preceding article, let G represent the force of gravity along the inclined plane CG, *g* the force by which the motion of C is actually accelerated in this direction, *k* the force of gravity along the inclined plane C*f*, and *p* the additional force by which the motion of A is accelerated from the action of C, *l* the force of gravity along *ch*, and *q* the force added to this by the action of C. Let P be equal to $k + p$, and Q to $l + q$. The force which the body C loses in the direction CG by acting upon A in the direction CF

and

and generating the force $A \times p$ in the direction Cf is $A \times p \times \frac{Cm}{Cf}$ or $A \times p \times \frac{Cf}{CG}$; the force which C loses in the same direction CG by its action on B is $B \times q \times \frac{cn}{ch}$ or $B \times q \times \frac{ch}{cG}$; consequently $C \times g + A \times p \times \frac{Cf}{CG} + B \times q \times \frac{ch}{cG}$ is equal to $C \times G$. If $\dot{x}$, $\dot{y}$, and $\dot{z}$ represent the fluxions of the respective spaces described by C, A, and B, by their motions, $\dot{x}$ will be to $\dot{y}$ as CG to Cf, and $\dot{x}$ to $\dot{z}$ as cG to ch. Therefore $Cg\dot{x} + Ap\dot{y} + Bq\dot{z}$ will be equal to $C\dot{x}G$, or $Cg\dot{x} + AP\dot{y} + BQ\dot{z}$ equal to $C\dot{x}G + Ak\dot{y} + Bl\dot{z}$. But if V, u, and v represent the respective velocities that are acquired by C, A, and B at the points G, f, and h, and I, K, L the respective velocities which the same bodies would have acquired by falling freely from the same altitudes from which they have descended; then (art. 434) $g\dot{x}$, $P\dot{y}$, and $Q\dot{z}$ will represent the respective fluxions of $\frac{1}{2}VV$, $\frac{1}{2}uu$, $\frac{1}{2}vv$; and $G\dot{x}$, $k\dot{y}$, $l\dot{z}$ will represent the fluxions of $\frac{1}{2}II$, $\frac{1}{2}KK$, and $\frac{1}{2}LL$. Therefore since we suppose these velocities to begin to be generated together, $CVV + Auu + Bvv$ is equal to $CII + AKK + BLL$, where AKK or BLL are to be subducted if A or B ascend while C descends. It is obvious that if we suppose the body C to act upon any number of bodies, or these to act on other bodies in any directions, the theorem will still obtain by collecting the sums of the products of all the bodies that act upon each other multiplied by the squares of their velocities.

526. If we suppose the bodies C, A, and B to ascend from their respective places G, f, and h with the motions which they have acquired, so as to be retarded by their gravity only, their common centre of gravity will rise to the same level from which it descended. For suppose X, Y, and Z to be the respective altitudes to which these bodies would rise in this manner, H the altitude which would be described by their common centre of gravity, h the altitude which it described in descending, I, K, and L the respective altitudes from which the bodies descended, and S the sum of the bodies; then by the last article $CX + AY + BZ$ will be equal to $CI + AK + BL$: and these

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sums are respectively equal to S × H and S × *h*, by art. 509; therefore H is equal to *h*. These theorems extend equally to bodies of all kinds, those that are void of elasticity, as well as those that have any degree of elasticity, there being no relative velocity generated by C in the directions in which it acts upon the other bodies. Whereas the theorems demonstrated in art. 517 and 519, concerning the equality of the sums of the products of the bodies multiplied by the squares of their velocities compared together before and after their collisions, extend only to such bodies as have a perfect elasticity. These last are founded on the equality of the relative velocities of C, and the several bodies A, B, *&c.* in their respective directions before and after the stroke; but those on art. 434, and the general principle described in art. 511. There may be an analogy however between those theorems, that may be explained perhaps from the motions which are generated in bodies by the actions of springs; but we are not to extend those theorems to motions of all kinds for the sake of this analogy.

527. For if the body C descend from any height IC before it begin to act upon the other bodies; or if there be any collision of the bodies while they descend, and they have no elasticity, or an imperfect one; or, in general, if there be any sudden communication of motion from one body to another, and the relative velocities in their respective directions be less immediately after that action than before it; in those cases the sum of the products of the bodies multiplied by the squares of their velocities will be less than it would have been if the bodies had descended freely from the same respective altitudes; and if the bodies be supposed to ascend with their respective velocities at any time, and their motions be retarded by their gravity only, the common centre of gravity will not ascend to the same level from which it descended. When the bodies C and A (*fig.* 235) that have no elasticity, or an imperfect one, suspended by equal lines KC and LA from the points K and L (that are on the same level, and at a distance from each other equal to the sum of the semidiameters of the bodies), after describing the arks IC and EA, strike one another; or when any body C after its descent from I is loaded with a new body at C

which it carries along with it in its ascent (as in a known experiment made by Mr. *Graham*), it is obvious that the ascent of their common centre of gravity must be less than its descent. To give another simple instance; suppose that the body C (*fig*. 236) descends in the vertical CD, and at the same time draws any body A along the horizontal line KL without friction by a line or chain CMA (which we suppose to be void of gravity) that is directed by the pully M, so that MA is always horizontal. First, let C draw the body A as soon as it begins to descend; and the accelerating force being always as the absolute force directly, and the matter that is to be moved inversely, the motion of each body will be accelerated by a force that is less than the accelerating force of gravity in the ratio of C to C + A. Let CG be equal to A*a*, and the square of the velocity of C when it comes to G, or of A when it comes to *a*, will be to the square of the velocity which C would have acquired by falling freely from C to G in the same ratio of C to C + A, the squares of the velocities acquired by descending from the same altitude being as the forces that generate them, when these forces act uniformly, by art. 434; consequently the product of the sum of C and A multiplied by the square of their common velocity is equal to the product of C multiplied by the square of the velocity which it would have acquired by the same perpendicular descent, if it had fallen freely from C to G; and if the bodies C and A be supposed to ascend from G and *a* with the respective motions acquired at these points, their common centre of gravity will rise to the same level from which it descended in this case. But let us suppose now that the body C first descends from M to C, and that the line or chain AMC is not stretched till it come to C, so that no motion is communicated to the body A till that instant; the motion acquired by C will be then divided betwixt C and A so as to produce equal velocities in each; the sum of the products of the bodies multiplied by the squares of these velocities will be less in this case than the product of C multiplied by the square of the velocity which it acquired by descending freely from M to C in proportion as C is less than C + A; and if the bodies C and A be supposed to ascend with those velocities from their respective places, the

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ascent of their centre of gravity will be less than its descent in the same ratio (*fig.* 230). If we suppose in art. 521 and 522, that the body C falls from I to C before it act upon A and B, and thereafter descends impelling those bodies by its gravity as above, let C*e* be to CI, as G*d* is to CD, and the sum of the products of the bodies multiplied by the squares of the respective velocities which they acquire when C comes to G, will be equal to the product of C multiplied by the square of that velocity only which it would acquire by descending freely from *e* to *d*. In the same manner it may be shown, that in the case of art. 524 (*fig.* 233), if C fall from any altitude before it act upon A and B, and thereafter descend impelling them as in that article, and the bodies be supposed to ascend with the respective velocities acquired by them from their respective places at any time, the ascent of the centre of gravity will be less than its descent. We have mentioned these instances, though they are obvious, to prevent mistakes from the expressions of some celebrated authors, who seem to represent this principle concerning the equality of the ascent and descent of the centre of gravity as general.

528. When the body C (*fig.* 236) was supposed to draw the body A along KL by the line or chain CMA from the beginning of its descent, a greater quantity of motion was generated in C and A by the uniform power of gravity acting upon C than that which C alone would have acquired by the same perpendicular descent CG, in the same proportion that the time of descent in CG is prolonged in the former case above what it is in the latter; and the like may be said of those cases which were described in art. 521 and 522, if regard be had to the directions in which the bodies move. And as the same power acting with the same direction upon the same body may be reasonably supposed to generate a greater force in a greater time, as well as a greater quantity of motion; so there is no ground to alter the usual manner of measuring the forces of bodies in motion on account of the preceding theorems, or of those that follow concerning the sums of the products of the bodies multiplied by the squares of their velocities. If we were to measure the forces of bodies in motion by the product of their quantities of matter, and of the squares of their velocities, the sum of the

forces acquired by the bodies C and A at G and *a* would be equal to the force which C alone would acquire by the same descent CG; and the same force that imprinted on the body C alone would cause it to ascend in the vertical from G to C, if it was imprinted on the bodies C and A at once, so as to generate equal velocities in the body A from A towards *a* along the horizontal line LK, and in the body C upwards from G, it would cause the body C to rise to the same height from G to C as in the other case, and at the same time cause the body A to describe *a*A equal to GC along the horizontal LK; the force which would be sufficient to produce those two effects would be always the same how great soever we should suppose the body A to be: and if we should likewise admit that the force which causes a given body C to ascend from G to C, and describe a given altitude GC, is always the same without regard to the time, it would thence follow that the motion of the body A from A to *a* is an effect that ought to be held of no account. An observation of the same kind might be made in other instances; but there is no necessity for perplexing the theory of motion with the consequences that follow from this doctrine concerning the mensuration of the forces of bodies; and therefore we proceed to argue from the principle in art. 511, which is universally allowed.

529. Hitherto we have supposed the body C (*fig.* 237) to act immediately by contact on the other bodies. Let the bodies A and B be now fixed to the axis KIL at the respective distances KA and LB, and the body C impinge on the inflexible lever IC (that is fixed perpendicularly to the same axis) with a direction and velocity represented by CD; and supposing the figure to be at rest before the stroke, let it be moveable about the axis KL only. Let CQ perpendicular to IC meet DQ parallel to it in Q; divide CQ in N, so that CN may be to NQ as the product of the body C multiplied by the square of its distance from the axis of motion to the sum of the products of the other bodies multiplied by the squares of their respective distances from the same axis; and DG parallel to CQ will intersect NG parallel to IC in G, so that CG will represent the velocity of C after the stroke; and if A*f* and B*h* be to CN as KA and

and LB to IC respectively, Af and Bh will represent the respective velocities of A and B after the stroke when there is no elasticity. For if we suppose any line CN to represent the velocity of C after the stroke in the direction CQ, then (because when the figure moves about the axis KL the velocity of any point A is to the velocity of any point C as KA to IC, or as Af to CN) Af will represent the velocity, and $A \times Af$ the motion of A. The motion which C must lose in the direction CQ by generating in A this motion $A \times Af$ must be to $A \times Af$ as KA to IC (by the principles of mechanics), or to $A \times CN$ as KA^2 to KC^2, and the motion which C loses in the same direction by producing in B a motion $B \times Bh$ is in like manner to $B \times Bh$ as LB to IC or to $B \times CN$ as LB^2 to IC^2. And the whole motion lost by C in the direction CQ being $C \times NQ$, it follows that $C \times NQ \times IC^2$ is equal to $A \times CN \times KA^2 + B \times CN \times LB^2$, and that CN is to NQ as $C \times IC^2$ to $A \times KA^2 + B \times LB^2$. Therefore since CQ was divided in this ratio in N, and the motion QD is not affected by the stroke, CG will represent the direction and velocity of C after the stroke. When C is perfectly elastic, produce DG till Dg be equal to 2DG; then Cg will show the direction and measure the velocity of C after the stroke, and the respective velocities of A and B will be represented by $2Af$ and $2Bh$.

530. When the body C and the lever are supposed to have no elasticity, the sum of the products of the bodies multiplied by the squares of their velocities after the stroke is less than the product of C multiplied by the square of its incident velocity in the ratio of Cd to CD, Gd being perpendicular to CD; but when C is perfectly elastic these are equal to each other. For by what was shown in the last article $C \times CN \times NQ$ is equal to to $A \times Af^2 + B \times Bh^2$, and by adding $C \times CG^2$, the whole sum becomes equal to the product of C by $CD^2 - CQ^2 + QCN$ or $CD^2 - CQN$, or (because DG or QN is to Dd as CD to CQ) $CD^2 - CD \times Dd$, that is, to $C \times CD \times Cd$; which is less than $C \times CD^2$ in the ratio of Cd to CD. But $C \times Cg^2 + A \times 4Af^2 + B \times 4Bh^2$ is equal to $C \times CD^2$; for let gn be perpendicular to CN in n, and that sum being equal to $C \times Cg^2 + C \times 4QNC$ (by what was shown in the last article) or to

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the product of C by Cn^2+gn^2+4QNC, it is equal (*elem.* 8, 2, QN and Nn being equal) to $C\times CD^2$. Therefore if we suppose that C acquires its incident velocity CD by falling from any altitude cC, and the bodies be supposed to ascend with their respective velocities immediately after the collision, so that their motions be retarded by their gravity only, their centre of gravity will ascend to the same height from which it descended in the latter case when C is supposed to have a perfect elasticity; but in the former case the ascent of the centre of gravity will be less than its descent in the same ratio as Cd is less than CD. These theorems are easily extended to the cases when several bodies strike the lever IC at once, or different levers fixed to the same axis with given directions and velocities; and when the elasticity is imperfect, the ascent of the centre of gravity will be always less than its descent, the motions of the bodies being supposed to be converted upwards after the collision.

531. Suppose now that the body C acts by its gravity only upon the lever IC, and by means of this lever impels the whole figure about the axis KL, the bodies A and B being supposed to have no gravity, then the accelerating force and the direction of gravity being represented by CD, the force and direction with which C will begin to descend will be represented by CG if the body C be allowed to slide along the lever IC, but by CN, if the body C be fixed to the lever, the force NG being destroyed in this case by the resistance of the axis. Because $C\times CG^2+A\times Af^2+B\times Bh^2$ is equal to $C\times CD\times Cd$, it follows that in either case the sum of the products of the bodies multiplied by the squares of their respective velocities is equal to the product of C multiplied by the square of the velocity which it would have acquired by the same perpendicular descent, if it had fallen freely in the vertical CD, providing the body C act upon the lever from the beginning of its descent (*fig.* 237, *N.* 2). It follows likewise from art. 529, that if I be the centre of gravity of the bodies A, B, &c. and a weight P act upon the lever IA at the distance IC from the axis of motion which we suppose to pass through I, then the force CG with which P descends will be to its gravity CD as $P\times IC^2$ to the sum of the products of the bodies A, B, &c. multiplied by the squares of their respective

tive distances from the axis added to $P \times IC^2$; and hence the motion of P may be determined when it turns the figure around the centre of gravity I by means of a rope PCZR that goes round the axis CZR.

532. If we suppose all the bodies C, A, and B (*fig.* 238) fixed to the axis at their respective distances CI, AK, and BL to gravitate in parallel lines CD, Aa, and Bb; let these lines be equal to each other, and represent the accelerating force of gravity; let Cg, Af, and Bh represent the forces by which their motions are actually accelerated with their respective directions perpendicular to IC, KA, and LB, while the figure moves upon its axis KL. Let DQ, am, and bk be perpendicular to those directions in Q, m, and k; and gd, fn, hr be perpendicular to CD, Aa, and Bb in d, n, and r. If CQ be greater than Cg, but Aa less than An, and Bb less than Br, then the body C loses by its action on the lever IC a force $C \times gQ$, and thereby the bodies A and B acquire the forces $A \times mf$ and $B \times kh$ respectively. Hence regard being had to the lengths of the several levers CI, KA, and BL, according to the known principles of mechanics, $C \times gQ \times IC$ will be equal to $A \times mf \times KA + B \times kh \times LB$, or (because the velocities Cg, Af, and Bh are as the distances from the axis IC, AK, and BL) $C \times Cg \times gQ$ equal to $A \times Af \times mf + B \times Bh \times kh$, that is, $C \times CQ \times Cg - C \times Cg^2$ equal to $A \times Af^2 - A \times Af \times Am + B \times Bh^2 - B \times Bh \times Bk$. But $CQ \times Cg$ is equal to $CD \times Cd$, $Af \times Am$ to $Aa \times An$ and $Bh \times Bk$ to $Bb \times Br$. Therefore $C \times Cg^2 + A \times Af^2 + B \times Bh^2$ is equal to $C \times CD \times Cd + A \times CD \times An + B \times CD \times Br$; from which it follows (by art. 434), that when all the bodies descend while the axis moves, the sum of the products of the bodies multiplied by the squares of the respective velocities acquired by them at any time is the same as if they had fallen freely along the perpendicular altitudes from which they have descended. But if any body as B (for example) had been on the other side of the axis KL so as to have ascended while the common centre of gravity of the bodies descended, hk had been equal to $Bk + Bh$. In this case the term $B \times CD \times Br$ must be subducted in the latter part of the last equation; and in general the sum of the products of the bodies multiplied by

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the squares of their respective velocities is equal to the difference of the sum of the products of those that descend multiplied by the squares of the velocities that would have been acquired by the same descents if they had fallen freely, and of the sum of the products of those that ascend multiplied by the squares of the respective velocities that would be acquired by falling freely along the respective altitudes to which they have arisen. In either case it follows that if the bodies be supposed to ascend from their respective places at any time, and to be retarded by their gravity only, their common centre of gravity will always ascend to the same level from which it descended. This principle is demonstrated in like manner when the bodies C, A, B, *&c.* are supposed to act upon one another by compound levers or other mechanical engines, without friction or resistance from the ambient medium. But it will not hold if we suppose any body first to impinge on the lever or engine with any assignable velocity, and then to descend with it.

533. It was advanced long ago by Mr. *Huygens** as a general principle, "That if bodies begin to move by their gravity, their common centre of gravity can never rise higher "than where it was at the beginning of the motion." To which he added as a second *hypothesis*, "That abstracting from the "resistance of the air and such obvious impediments, a compound "pendulum will describe equal arks in its descent and ascent." And by these two principles he was able to determine the length of a simple pendulum that should vibrate in a void in the same time with a compound one in any similar arks, and to find the centre of oscillation of bodies. He did not then affirm that the centre of gravity of the bodies would always rise to the same height from which it descended, but that it will never rise to a greater height than this; which is indeed a general principle, for the ascent of the centre of gravity will be always found to be either equal to its descent or less than it, but never greater. He seems however to go farther afterwards, and to affirm that

* Horol. Oscil. par. 4. Hyp. 1. & 2.

bodies

bodies always retain their *vis ascendens* *, as he calls it, by which their centre of gravity would rise to the same level from which it descended. This principle obtains indeed in all the cases he has mentioned (these being called hard bodies by him which are supposed to have a perfect elasticity) and in many others; as has been shown in the preceding articles; where we have endeavoured to distinguish those cases in which this principle takes place from those wherein it cannot be admitted, and to show at the same time that no useful conclusion in mechanics is affected by the disputes concerning the mensuration of the force of bodies in motion which have been objected to mathematicians †.

534. Suppose therefore OV (*fig.* 238) to be equal to the length of a simple pendulum that in a void performs its vibrations in similar arks in the same time with the compound pendulum described in art. 532, or let OV be the distance of a point in this latter pendulum that moves in it with the same velocity as if OV was a simple pendulum suspended at O. Let S represent the sum of the bodies C, A, and B, and OG be the distance of their centre of gravity from the axis. While the pendulum moves, let the points C, B, A, G, and V descend to *c*, *b*, *a*, *g*, and *v* respectively; let GM be the perpendicular descent of the centre of gravity, and VR the perpendicular descent of the point V. Then because the velocities of the points C, B, A, G, and V are as their distances from the axis of oscillation, and the velocity acquired at *v* is such as would cause a body to ascend from R to V (by the supposition), and the altitudes to which bodies would ascend by the velocities acquired at *c*, *b*, *a*, and *v* are in the duplicate ratio of these velocities, it follows that C, B, and A would ascend by their respective velocities at those points to the altitudes $VR \times \frac{IC^2}{OV^2}$, $VR \times \frac{LB^2}{OV^2}$ and $VR \times \frac{KA^2}{OV^2}$. And since their common centre of gravity would ascend to the same

* *Hæc constans lex est corpora servare* vim suam ascendentem, *& idcirco summam quadratorum velocitatum illorum semper manere eandem. Hoc autem non solum obtinet in ponderibus pendulorum & percussione corporum durorum, sed in multis quoque aliis mechanicis experimentis.* Observ. D. *Huygen*, in literis D. March de l'Hospital, *&c.* Oper. Vol. I. p. 258.

† *Analyst. Query* 9.

same altitude from which it descended, by art. 532, it follows (art. 509), that $C \times VR \times \frac{IC^2}{OV^2} + B \times VR \times \frac{LB^2}{OV^2} + A \times VR \times \frac{KA^2}{OV^2}$ is equal to $S \times GM$. But the arks described by G and V being similar, GM is to VR as OG to OV; consequently $S \times OG \times OV$ is equal to $C \times IC^2 + R \times LB^2 + A \times KA^2$; and OV is found by multiplying each body by the square of its distance from the axis of oscillation, and dividing the sum of the products by $S \times OG$, which is the product of the sum of the bodies multiplied by the distance of their common centre of gravity from the same axis. The same demonstration being applicable to any number of bodies, we may conclude that when any body moves about a given axis, the distance of its centre of oscillation from this axis (or the length of a simple pendulum that vibrates in a void in the same time with the body in similar arks) is found by computing the fluent when each particle or element of the body is supposed to be multiplied by the square of its distance from the axis, and dividing this fluent by the product of the body multiplied by the distance of its centre of gravity from the same axis.

535. If the points C, A, and B be in one plane that is perpendicular to the axis of oscillation in O, let Cc, Bl and Ak be perpendicular to OG in i, l and k; then OC^2 being equal to $OG^2 + CG^2 - 2OGi$, OB^2 to $OG^2 + BG^2 + 2OGl$ and OA^2 to $OG^2 + AG^2 + 2OGk$ (*elem.* 12 and 13, 2), the point i being betwixt O and G, and the points l and k on the other side of G, and $C \times iG$ being equal to $B \times lG + A \times kG$ (art. 509), it follows that the sum of the products of the bodies multiplied by the squares of their distances from O, or $S \times OG \times OV$ is equal to $S \times OG^2 + C \times CG^2 + B \times BG^2 + A \times AG^2$; consequently $S \times OG \times GV$ is equal to the sum of the products when each body is multiplied by the square of its distance from the centre of gravity, and GV the distance of the centre of oscillation from the centre of gravity is found by dividing this sum by $S \times OG$; whence the computation of the distance of the centre of oscillation from the axis in solids is in some cases abridged.

536. To

536. To give an example, let DEd(*fig.* 239) be the section of a sphere through its centre G by a plane perpendicular to the axis of oscillation, Dd the diameter of this circle perpendicular to the axis, GE the radius perpendicular to Dd, and PFp any concentric circle; let MP an ordinate perpendicular to Dd at P meet DEd in M, MN be perpendicular to GE in N, and the ratio of n to 1 express that of the circumference of a circle to its radius. Let a cylindric surface be imagined to stand on the circumference PFp perpendicular to the plane DEd, and terminated by the surface of the sphere, and its altitude being 2PM, it may be expressed by $2n \times GP \times GN$, which being multiplied by the square of GP (which is the distance of the particles in each section of this surface perpendicular to the axis of oscillation from the centre of gravity of the section), and the product $2n \times GN \times GP^3$ being multiplied by the fluxion of GP, or (because GP^2 is equal to $GD^2 - GN^2$, and the fluxion of GP is to the fluxion of GN as GN to GP) the product of $2n \times GN^2$ and $GE^2 - GN^2$ being multiplied by the fluxion of PM, the fluent by the converse of art. 146, will be the product of $2n \times GN^3$ by $\frac{1}{3} GE^2 - \frac{1}{5} GN^2$. But this fluent becomes equal to $\frac{4}{15} \times n \times GE^5$ when P has described the whole radius DG, and GN becomes equal to GE; and this being divided by $\frac{2}{3}\ n \times GE^3 \times OG$ (which expresses the solid content of the sphere multiplied by OG, by what was shown in the Introduction), GV the distance of the centre of oscillation from the centre of gravity in the sphere is found to be $\frac{2}{5}\frac{GE^2}{OG}$ or to be two fifths of a third proportional to OG and GE. This subject having been treated of fully in the *Horol. Oscil. par.* 4, *Acta Lipsiæ*, 1714, and *Method. Increm. prop.* 24, and in several other pieces, we shall not insist on it further here, and shall only add, that when a weight P (*fig.* 237. *N.* 2) turns a figure about its centre of gravity I by means of a rope PCZR that goes round the axis CZR as in art. 531, let V be the centre of oscillation of the figure when C is the centre of suspension, let S denote the mass or weight of the figure to be moved, Ie be taken upon IC in the same ratio to IC as P is to S; then CG the force by which the motion of P will be actually accelerated will be

to

to CD the accelerating force of gravity as eI to eV. For (by art. 531) CG is to GD in this case as $P \times IC^2$ to $A \times AI^2 + B \times BI^2$, &c. which last is equal to $S \times IC \times IV$, by art. 535; consequently CG is to GD as $P \times IC$ to $S \times IV$, or as eI to IV, and CG to CD as eI to eV; which agrees with the solution given by the learned Mr. *Daniel Bernoulli*, *Comment. Petropolit. tom.*

537. Sir *Isaac Newton* has considered the motion of water issuing from a cylindric vessel ABDC (*fig.* 240. *N.* 1) at an orifice EF in the bottom CD, *Princip. lib.* 2, *prop.* 36. His doctrine on this subject may receive some illustration from the following considerations. While the water issues at the orifice EF, that which remains in the vessel subsides at the same time; and though the particles of this water descend with unequal velocities, we may consider the velocity with which the surface AB descends to be their mean velocity. This velocity manifestly begins from nothing (as that of any heavy body that descends by its gravity), and while it is accelerated is always to the velocity with which the water issues at EF in the ratio of EF to AB. The continual effect of the gravitation of the whole mass of water may be considered as threefold. It accelerates, for some time at least, the motion with which the water in the vessel descends; it generates the excess of the motion with which the water issues at the orifice above the motion which it would have had in common with the rest of the water; and it acts on the bottom of the vessel at the same time. Let the velocity with which the water issues at EF at any term of the time be represented by X, the velocity with which the surface AB subsides by V, the accelerating force of gravity by g, the force which would generate the acceleration of V by f, and the time from the beginning of the motion by T. The gravitation of the whole mass of water in the cylindric vessel ABCD may be expressed by $AB \times AC \times g$; and because the force f is employed in generating the acceleration of the motion with which the water subsides in the vessel, the force $AB \times AC \times \overline{g-f}$ is what we are to suppose to be employed in acting upon the bottom, and in generating the velocity $X - V$ in the water that issues at the orifice. Suppose that the ratio of $r-1$ to 1 expresses the proportion

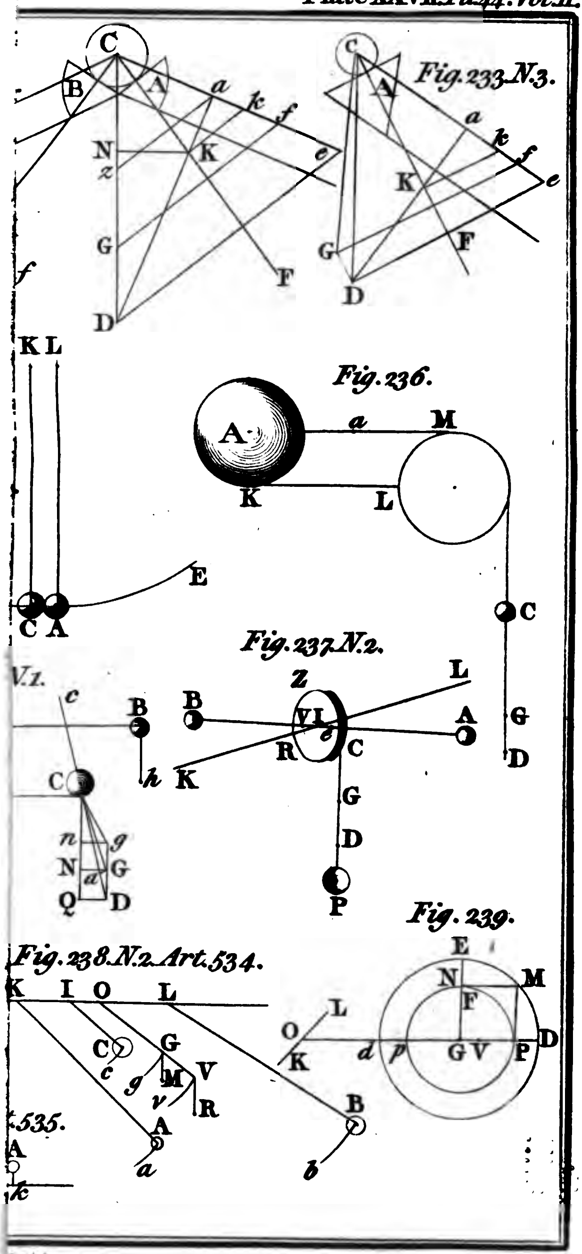
Fig. 233. N. 3.
Fig. 236.
Fig. 237. N. 2.
N. 1.
Fig. 238. N. 2. Art. 534.
Fig. 239.
535.

proportion of the parts of this last force which produce these two effects, or that r is to 1 as the force $AB \times AC \times \overline{g-f}$ is to that part of it which we conceive to produce the velocity $X-V$ in the water issuing at EF; and this part will be expressed by $AB \times AC \times \frac{g-f}{r}$. The quantity of water which would issue at the orifice EF in any time T with the velocity X continued uniformly is expressed by $EF \times T \times X$, and the force which would generate the velocity $X-V$ in this quantity of water is as the quantity of motion that would be generated in this manner (or $EF \times TX \times \overline{X-V}$) directly, and the time T inversely, that is as $EF \times X \times \overline{X-V}$, or (because X is to V as AB to EF, and $X-V$ to X as $AB-EF$ to AB), as $EF \times \frac{AB-EF}{AB} \times XX$, which we are to suppose equal to $AB \times AC \times \frac{g-f}{r}$ that represents the same force. The square of the velocity that would be acquired by the descent AC is expressed by $2AC \times g$ (art. 434), and if KC be to AC as $AB \times AB$ is to $2r \times EF \times \overline{AB-EF}$, and A denote the velocity that would be acquired by the descent KC, then AA will be to $2AC \times g$ as KC to AC, or as $AB \times AB$ to $2r \times EF \times \overline{AB-EF}$, and consequently as XX is to $2AC \times \overline{g-f}$. Therefore AA will be to XX as g is to $g-f$, and g to f as AA to $AA-XX$. From this it follows, that if we suppose the fluxion of X (or of V which is in a given ratio to X) to vanish, in order to find its greatest value, or the limit of all its values, by art. 242, f the force which accelerates V must vanish, $g-f$ must be equal to g, and X to A. In general, the descent by which any velocity X of the issuing water would be acquired is to KC the descent by which A would be acquired as $g-f$ to g. It appears also that the fluxion of V is to the fluxion of the velocity of a body that descends freely at the same time in the vertical as f to g, or as $AA-XX$ to AA. If the fluxions of T, X, and V be represented by t, x, and v respectively, then f will be represented by $\frac{v}{t}$, and AA will be to $AA-XX$ as gt to ft or gt to v, or as $AB \times gt$ to $EF \times x$, because v is to x as EF to AB, by art.

art. 24, so that t is expressed by a quantity that is to x in the compound ratio of AA to AA—XX, and of AB to EB. By AB and EF we mean the areas of the section of the cylinder and of the orifice, and not their diameters. We shall suppose the ratio of r to 1 to be invariable; and because different suppositions have been made concerning this ratio, we shall show how the motion of the water may be computed from any of them, before we enquire what this ratio is.

538. Let anl (*fig.* 240, N. 1 and 2) be an equilateral hyperbola, c the centre, a the vertex, ab perpendicular to ca meet the asymptote in b, ab represent the velocity A, ad the velocity X, join cd and produce it till it meet the hyperbola in n; then the time T in which the velocity of the issuing water becomes equal to X will be to the time in which a body would fall from K to C by its gravity in the compound ratio of the area of the orifice EF to the area of the bottom CD or AB, and of the hyperbolic sector can to the triangle cab, if the water be supposed to be supplied at the surface AB so that the vessel be kept always full. For let nm be perpendicular to the axis ca in m, and ad^2 will be to nm^2 as ca^2 to cm^2 or ca^2+nm^2 (by the property of the hyperbola), and cm^2 to ca^2 as ab^2 to $ab^2—ad^2$, or as AA to AA—XX. The fluxion of the sector can is to the fluxion of the triangle cad (or $\frac{1}{2}$ A$\dot{x}$) as cn^2 to cd^2 (art. 120), or cm^2 to ca^2, and therefore as AA to AA—XX, or as AB$\times g\dot{t}$ to EF$\times\dot{x}$; consequently the fluxion of the sector can is to A$g\dot{t}$ as AB to 2EF, and the sector can to AgT in the same ratio by art. 24. Let Z express the time in which a body falls from K to C, and by its descent acquires the velocity A, then Z will be expressed by $\frac{A}{g}$, and T will be to Z as $can\times 2$EF to ca^2 $\times$AB. Hence if Z, T, $ab\times\frac{AB}{EF}$ and L be proportional, and the ratio of which L is the logarithm (the *modulus* being ab) be that of c to d, then the velocity ab will be to the velocity acquired at the end of the time T as $c+d$ to $c—d$. For example, if the area AB be to the area EF as 10 to 1, and T be supposed equal to Z, L will be to ab as 20 to 1, c to d in a greater ratio than 485000000 to 1; and the excess of ab above

ad

ad will be less than the 242000000 part of *ab* in the time a body would fall from K to C, though according to this theory *ad* can never become precisely equal to *ab*.

539. Let *mk* perpendicular to the asymptote meet the hyperbola in *p*, join *cp*, and the quantity of water that issues at the orifice EF in the time T will be to the quantity that would have issued at EF in the same time if the velocity had been always equal to A as the hyperbolic sector *cap* is to the sector *can*. For let the quantity of water that issues at EF in the time T be represented by Q, and its fluxion by *q*, then *q* will be expressed by $Xt \times EF$, which (by art. 537) is to $EF \times \frac{Xx}{g}$ as $AA \times EF$ to $\overline{AA-XX} \times AB$; so that the fluent of Xt is to half the logarithm of the ratio of AA—XX to AA as EF to $g \times AB$, the *modulus* being AA. Let *ae* be perpendicular to the asymptote in *e*, and if the *modulus* be ce^2 or $\frac{1}{2}$ AA, the area *aekp* will be the logarithm of the ratio of *pk* to *ae*, or of *ce* to *ck* or of *ca* to *cm*, that is, *aekp* will be equal to one half of the logarithm of the ratio of ca^2 to cm^2, or of AA—XX to AA. Therefore the fluent of Xt is to the area *aekp*, or the sector *cap*, as 2EF to $g \times AB$. But $A \times T \times EF$ expresses the quantity of water that would have issued at the orifice EF in the same time T with the velocity A, and $A \times T$ is to the sector *can* in the same ratio, by the last article. Therefore Q is to $A \times T \times EF$ as the sector *cap* to *can*. Hence the difference of $AT \times EF$ and Q is to $AT \times EF$ as the sector *cpn* to *can*, and is to the quantity that would issue at the orifice EF in the time Z (in which a body would fall from K to C) as $EF \times ncp$ to $AB \times cab$. Let *ch* be taken on the asymptote equal to 2*ce*, and *hf* perpendicular to *ch* meet the hyperbola in *f*, let *mn* produced meet the asymptote in *u*, and *nl* be perpendicular to the asymptote in *l*; then *cl* being always less than *cu* or 2*ck*, it follows that *ncp* is always less than the sector *caf* or the hyperbolic logarithm of 2; and that the difference of the quantity of water which issues at the orifice EF, and that which would have issued in the same time with the velocity A, is to what would issue with the velocity A in the time Z in a ratio that is always less than

than that of $EF \times caf$ to $AB \times cab$, but that continually approaches to this ratio as its limit.

540. In the two preceding articles we supposed the vessel to be kept always full to the altitude AC (*fig.* 241), and the water to be always supplied at the surface AB with the velocity V with which the water in the vessel subsides. If we now suppose that no water is supplied, but that the upper surface AB subsides while the water issues at the orifice EF, let aC be the altitude of the water at the beginning of the motion, AC its altitude after any time, and let the ratio of e to 1 be that which is compounded of the ratio of $2r$ to 1 and of AB—EF to EF. Let a series of continued proportionals be formed, of which Ca and CA are the first terms, and CH be the term whose place in the progression is denoted by $e+1$ when e is any rational number, or more generally let the logarithm of CH be to the logarithm of CA as e is to 1, the *modulus* being Ca; then the velocity of the water issuing at EF will be such as would be acquired by a descent that is to AH in the invariable ratio of AB^2 to $EF^2 \times \overline{e-1}$. For let AC be represented by H and its fluxion by $-h$ (which is negative because AC decreases), let D represent the descent by which X would be acquired and d its fluxion, then since $-h$ is represented by Vt, gd by Xx (art. 434), v is to x as EF to AB, and g is to f as AA to AA—XX; it follows that $-h$ is to d as $EF^2 \times AA$ to $AB^2 \times \overline{AA-XX}$, or as $EF^2 \times H$ to $AB^2 \times H - EF^2 \times e \times D$, so that H$h$ $\times \frac{AB^2}{EF^2}$ is equal to $eDh - Hd$; and hence by multiplying by H^{-e-1} and finding the fluents (by the converse of art. 99 and 168), $EF^2 \times \overline{e-1} \times D$ is equal to $AB^2 \times \overline{CA-CH}$ or $AB^2 \times \overline{CH-CA}$, according as e is greater or less than unit. But when e is equal to unit, then D will be found to be to CA in the compound ratio of AB^2 to EF^2, and of the logarithm of CA to the *modulus* Ca. When e exceeds unit the velocity of the water is greatest when CA is to Ca as unit is to the number the logarithm of which is to the logarithm of e as unit to $e-1$, and the velocity is such as would be acquired by a descent that is to AC as AB^2 to $e \times EF^2$.

541. Let

541. Let IG (*fig.* 240, *N.* 1) perpendicular to the area EF at G meet AB in H, and be to IH in the duplicate ratio of the area AB to the area EF; and let AMEFNB be such a cataract of water that any horizontal section of it as MN may be always inversely in the subduplicate ratio of IR its distance from I. Then supposing, with Sir *Isaac Newton*, the water around this cataract to be congealed, and the water to enter always into the cataract in the surface AB with the velocity that would be acquired by the descent IH, the water will descend in the form of this cataract the sections of which diminish in the same proportion as the velocity of the descending fluid increases, and will exert no pressure on the ambient congealed part. Thus, the water in the vessel is distinguished into two parts; the gravitation of the cataract generates the increase of the motion of the water that descends through every section, or the excess of that with which it issues at the orifice above what it had in entering the surface AB; while the gravitation of the ambient parts is what acts upon the bottom of the vessel. The ratio of these two parts is that of $2EF$ to $AB-EF$. For, since the section MN is inversely in the subduplicate ratio of IR, the solid AMEFNB is equal to $2EF \times IG - 2AB \times IH$ (as may be easily deduced from art. 307), which is to $2EF \times IG$ as $AB-EF$ to AB, because IH is to IG as EF^2 to AB^2. The content of the cylinder is $AB \times HG$, or $IG \times \frac{AB^2-EF^2}{AB}$; consequently the content of the cataract is to that of the cylinder as $2EF$ to $AB+EF$. Supposing therefore with Sir *Isaac Newton*, that the forces which generate the velocity $X-V$ in the water that issues at EF and that act upon the bottom of the vessel are the same when all the water is fluid, the ratio of r to 1 will be that of $AB+EF$ to $2EF$. And if we substitute this ratio for that of r to 1 in the preceding articles, A the limit of the velocities with which the water issues at EF (when the vessel is always kept full to the height CA) will be such as is acquired by the descent KC, if KC be to $\frac{1}{2}AC$ as $AB^2 \times 2EF$ to $EF \times \overline{AB^2-EF^2}$, or to AC as AB^2 to AB^2-EF^2, that is, if KC be equal to IG. The time in which any velocity X (or *ad*) is acquired, and the quantity of water that issues in that time, will be such

as were determined in art. 538 and 539, abstracting from friction, the resistance of the air, and the effect of the oblique motions of the particles described by Sir *Isaac Newton*, by which this quantity is diminished (*fig.* 241). If we substitute this value for the ratio of r to 1 in art. 540, where the water was not supposed to be supplied, we shall find e to 1 as $AB^2 - EF^2$ to EF^2, or $e + 1$ to 1 as AB^2 to EF^2; and if the logarithm of CH be to the logarithm of CA as $AB^2 - EF^2$ to EF^2, the *modulus* being Ca, the velocity of the water issuing at EF will be such as would be acquired by a descent that is to AH in the invariable ratio of AB^2 to $AB^2 - 2EF^2$. If we had supposed that the action on all parts of the area CD is the same, or that the force which generates the velocity X—V in the water issuing at EF is to the action on the bottom of the vessel (or 1 to $r - 1$) as the area EF to the area AB—EF, or 1 to r as EF to AB, then KC would have been to $\frac{1}{2}$ AC as AB to AB—EF, and e to 1 as $2AB \times \overline{AB - EF}$ to EF^2 (*fig.* 240, *N.* 1). We supposed that the forces which generate the motion X—V in the water that issues at EF and that act upon the bottom are in the same ratio when the water that is without the cataract AMNEFB is congealed, and when it is fluid; but there are several differences betwixt the motion of the water in these two cases. In the first the vein of water is no more contracted after its exit than the figure of the cataract requires; whereas in the latter case if the water issue at EF through a thin plate, the vein is immediately contracted after its exit in consequence of the oblique motions of the particles converging towards the orifice; and the area of a horizontal section of it at a little distance from the orifice is found to be less than the orifice in the ratio of 1 to $\sqrt{2}$ nearly when AB is much greater than EF; and the quantity of water that issues at EF is found to be nearly the same that would have issued in the same time if the ratio of r to 1 had been that of AB to EF according to the second hypothesis. If we suppose that in this case the quantity of water which issues at EF answers to the second hypothesis, but that the velocity answers to the first when we substitute the section of the vein of water after it is contracted for EF, then the area of the orifice EF will be to this section of the vein of water in the subduplicate ratio of

$AB^2 + \overline{AB-EF}^2$ to AB^2, which is always less than the ratio of $\sqrt{2}$ to 1, but is very near it when EF is very small compared with AB, and is a ratio of equality when AB and EF are equal. But when the water issues not at EF through a very thin plate, or when the vessel is not cylindric, the motion of the water and form of the vein is different. See on this subject *Princip. lib.* 2, *p.* 329, *Edit.* 3.

542. When the water is supposed to be supplied in a cylinder, so as to stand always at the same altitude above the orifice, there is an analogy between the acceleration of the motion of the water that issues at the orifice and the acceleration of a body that descends by its gravity in a medium which resists in the duplicate ratio of the velocity of the body, that deserves to be mentioned. Let g represent the force of gravity, R the resistance of the medium when the velocity is X, and let R be to g as XX to AA; then g—R the force by which the motion of the body is actually accelerated in its descent will be to g as AA—XX to AA, and A will be the greatest velocity which the descending body can acquire, or (to speak more accurately) the limit of all its possible velocities, because if X be supposed equal to A, R will be equal to g, and there can be no further acceleration. The fluxions of the velocity X and time T being represented by x and t, g—R will be expressed by $\frac{x}{t}$, and t will be to $\frac{x}{g}$ as AA to AA—XX. Hence if the resistance be equal to the force of gravity when the velocity is equal to that which would be acquired by the descent IG (or the limit of the velocities which the descending body can acquire, and the limit of the velocities with which the water issues at the orifice EF be equal), then T the time in which the descending body acquires any velocity X will be to T the time in which the water issuing at EF acquires the same velocity in the invariable ratio of AB to EF; because we found in art. 537, that t was to $\frac{x}{g}$ in the compound ratio of AA to AA—XX and of EF to AB; so that t is to t as AB to EF, and T to T in the same ratio.

543. In the same manner it appears, that if S be the space described by the body while it descends in such a medium in any

 time

time T, then the quantity of water that issues at the orifice EF in a time $T\times\frac{EF}{AB}$ will be equal to a cylindric column on the base EF of a height equal to $S\times\frac{EF}{AB}$. For since the times in which the body and the water acquire equal velocities are always in the invariable ratio of AB to EF, it follows that S the space described by the body in the time T is to the height of a column of water on the base EF equal to the quantity that issues at EF in the time T or $T\times\frac{EF}{AB}$ in the same ratio.

544. The same conclusions follow from the principles described above in art. 525 and 532, which are applied in an ingenious manner to this doctrine by Mr. *Daniel Bernouilli*, *Comment. Acad. Petrop. tom.* 2, who seems first to have determined rightly the manner in which the motion of water issuing from any vessel is accelerated, when we abstract from the impediments above mentioned. Supposing the surface AB of the fluid to subside in the vessel, and the fluxion of the time being represented by t, and that of the altitude AC by $-h$ as formerly, the fluxion of the square of the velocity of a body that descends freely in the vertical will be expressed by $-2gh$, the fluxion of the square of the velocity V with which the mass of water contained in the vessel actually descends by $-2fh$ (art. 434), and since the particle of water which issues at the orifice in the time t may be represented by $AB\times -h$, if we suppose $AB\times AC\times -2gh + AB\times AC\times 2fh$ equal to $AB\times -h \times \overline{XX-VV}$ (in consequence of what was shown in art. 525 and 532), it will follow that $2AC\times\overline{g-f}$ is equal to $XX-VV$, which is to XX as AB^2-EF^2 is to AB^2. Therefore if KC be to AC as AB^2 to AB^2-EF^2, and A be the velocity which would be acquired by the descent KC (so that AA may be to $2AC\times g$ in the same ratio), then $2AC\times\overline{g-f}$ will be to XX as $2AC\times g$ is to AA, and $g-f$ to g as XX to AA; which is agreeable to what we found in art. 537 and 541, in a different manner. And this is conformable to what was first taught by Sir *Isaac Newton*, that though the pressure upon EF is to the pressure upon

upon the base CD, before the orifice is opened, as the area EF to the area CD; yet when we suppose the water to issue at EF, and to have acquired its utmost velocity, the force that generates the velocity $X-V$ in the water at EF is measured by the gravity of the cataract AMEFNB, or by a column of the fluid of an altitude equal to $2HG \times \frac{AB}{AB+EF}$ on a base equal to the section of the vein of water after it is contracted; that is, the quantity of motion which is generated in the water issuing at EF with that uniform velocity, is equal to the motion which such a column of water would acquire by falling freely with its gravity in an equal time. He has not enquired into the manner in which the water is accelerated from the beginning of the motion; but if we represent the content of the cataract AMEFNB by C, and suppose $C \times \overline{g-f}$ equal to $EF \times X + \overline{X-V}$ the force which generates the velocity $X-V$ in the water issuing at EF, then, because C is to $2EF \times HG$ as AB to AB+EF, $X-V$ to X as AB—EF to AB, and AA is supposed to be to $2AC \times g$ as AB^2 to AB^2-EF^2, it will follow, that XX is to AA as $g-f$ is to g, as we found above.

545. The ratio of the action on the bottom of the vessel to the force that generates the velocity $X-V$ in the water issuing at EF (or that of $r-1$ to 1), which was deduced from the cataract after Sir *Isaac Newton*'s method in art. 541, follows likewise from the principle described in art. 525 or 532. Let P represent the first of these two forces, F the second, and P+F will be equal to $AB \times AC \times \overline{g-f}$ (by what was shown in art. 537), which is equal to $\frac{1}{2}AB \times \overline{XX-VV}$ or $\frac{1}{2}AB \times XX \times \frac{AB^2-EF^2}{AB^2}$ by what was deduced from that principle in the last article. But F is equal to $EF \times X \times \overline{X-V}$ (by art. 537), or $EF \times XX \times \frac{AB-EF}{AB}$; therefore P+F is to F (or r to 1) as $\frac{1}{2}AB \times \frac{AB^2-EF}{AB^2}$ to $EF \times \frac{AB-EF}{AB}$ or as AB + EF to 2EF; and P to F (or $r-1$ to 1) as AB—EF to 2EF, which is the same ratio that was deduced from the cataract in art. 541; and in cor. 2 and 5, prop. 36, Princip. lib. 2, where the water is supposed to have acquired its utmost velocity.

546. It must be acknowledged, however, that the preceding theory concerning the manner in which the water issuing at EF is accelerated from the beginning of the motion, is not to be considered as accurate in all respects, being founded on the hypothesis, that all the particles of the fluid within the cylindric vessel descend with the same velocity V, and that the water issuing at EF acquires the velocity X—V at once, which cannot be supposed to hold accurately. The acceleration of V is similar to that of a heavy body descending by its gravity in a medium that resists in the duplicate ratio of the velocity (the relative gravity of the body in the fluid being supposed equal to *g*) by what was shown in art. 542. And as the fluxion of the velocity of such a body is the same at the beginning of the descent, as if the body fell freely by the gravity *g*; so when the orifice EF is opened in the bottom of the vessel, if V or X be supposed to begin from nothing, AA—XX must be equal to AA at the beginning of the motion, and consequently *f* equal to *g*, so that the fluxion of V must be then equal to the fluxion of the velocity with which the water or any other body descends freely by its gravity. From which it follows, that, according to this theory, the pressure on the bottom of the vessel is wholly taken off at the instant of time when the water begins to issue at EF; and as this conclusion cannot be admitted, we may learn from this instance that this theory is not to be considered as perfectly exact. It will be worth while however to pursue this speculation a little further, and to show how the method described in art. 537 and 541 may be applied for determining the motion of water issuing from other vessels.

547 (*Fig.* 242). Suppose now the vessel to consist of two cylinders *abcd*, ABCD; and let *ab* the section of the upper part be greater than AB. The velocity of the water at EF being represented by X, and the velocity in the vessel ABCD by V, as formerly, let its velocity in *abcd* be represented by Z, and the forces by which V, Z, and X are accelerated by *f*, *p*, and F respectively. Let the sections AB and *ab* be represented by B and C, the altitudes AC and *ac* by *b* and *c* respectively, and the aperture EF by O; let the surface ACDB continued upwards intersect the plane *ab* in LM. Then the force that acts upon the surface CD corre-

sponding

sponding to that which is supposed (according to this method) to generate the velocity X—V in the water issuing at EF will be expressed, as in art. 537, by $rOX \times \overline{X-V}$, or (according to the ratio of r to 1 that was deduced from the cataract in art. 541) by $XX \times \frac{BB-OO}{2B}$. In like manner the force which generates the velocity V—Z at the surface AB is $OX \times \overline{V-Z}$, or (because V is to Z as C to B, and V to X as O to B) by $XX \times \frac{C-B}{C} \times \frac{OO}{B}$; and if this force be increased in the ratio of $ab+AB$ to 2AB (according to art. 541), or of C+B to 2B, we shall have $XX \times \frac{CC-BB}{2C} \times \frac{OO}{BB}$ for the action on the whole surface *cd* corresponding to that which generates the velocity V—Z in the water, while it passes from the upper into the lower cylinder at the surface AB. But because all the particles of the water that are in the same section of the vessel are supposed to descend with equal velocities in this theory, and to contribute equally to the actions of the fluid, we are to diminish this force in the ratio of AB to *ab*, or of B to C, that we may have the part of it $XX \times \frac{CC-BB}{2B} \times \frac{OO}{CC}$ which is to be ascribed to the column LCDM. Therefore since the velocity of the water in ACDB is accelerated by the force f, and its velocity in LABM by the force p, we are to suppose $AC \times AB \times \overline{g-f} + AL \times AB \times \overline{g-p}$, or $Bb \times \overline{g-f} + Bc \times \overline{g-p}$ equal to $XX \times \frac{BB-OO}{2B} + XX \times \frac{CC-BB}{2B} \times \frac{OO}{CC}$ or $XXB \times \frac{CC-OO}{CC}$, that is $\overline{b+c} \times g - bf - cp$ equal to $XX \times \frac{CC-OO}{CC}$; consequently if KC be to LC (or $b+c$) as ab^2 to ab^2-EF^2 or CC to CC—OO, and A denote the velocity that would be acquired by the descent KC, XX will be to AA as $\overline{b+c} \times g - bf - cp$ is to $\overline{b+c} \times g$, and A will be the limit of all the values of X. The velocities X, V, and Z, and their respective fluxions are in an invariable ratio, so that f will be to F as v to x, or V to X, or O to B; and p will be to F as Z to X or O to C. Therefore XX will be to AA as

$g - F \times \frac{O}{b+c} \times \overline{\frac{b}{B} + \frac{c}{C}}$ to g; or if LC be represented by H, and $\frac{bO}{B} + \frac{cO}{C}$ by K, XX will be to AA as $gH - FK$ to gH; consequently if the fluxion of X be represented by $\dot{x}$, and the fluxion of the time by $\dot{t}$, since $\dot{x}$ may be expressed by $F\dot{t}$, it follows that $\dot{t}$ will be expressed by $\frac{\dot{x}K}{gH} \times \frac{AA}{AA-XX}$. Hence if the velocity A be represented by *ab* (Fig. 240, n. 2), and any lesser velocity X by *ad*, and the water be always supplied at the surface *ab* with the velocity Z, the time in which the water issuing at EF will acquire the velocity X, will be to the time of descent from K to C in the compound ratio of the hyperbolic sector *can* to the triangle *cab* and of K to H. If we had supposed *r* to 1 as the area CD to the area EF (which was Sir *Isaac Newton's* hypothesis in the first edition of his *Principia*), then KC ought to have been taken in the same ratio to $\frac{1}{2}$LC as $1 - \frac{O}{B} + \frac{OO}{BB} \times \frac{C-B}{C}$ is to 1, and A being supposed equal to the velocity that would be acquired by the descent KC, the construction would have been in other respects the same.

548. When *ab* the uppermost section of the vessel and the area of the orifice EF with the altitude LC remain, the descent KC and the velocity A are the same, without any regard to the ratio of LA to AC. Hence if we suppose the water to be continually supplied into a cylinder LCDM at the surface LM, with a velocity that is less than V in any given ratio, let this ratio be that of LC or AB to *ab*, and if KC be to LC as ab^2 to $ab^2 - EF^2$, the utmost value of X will be the velocity that is required by the descent KC. And if the water be supposed to be always supplied at the surface LM, without having any velocity communicated to it (but what it receives from the water beneath, which cannot descend without it), then KC will be equal to LC; and the utmost velocity of the water at EF will be such as would be acquired by the descent LC, the altitude of the water in the vessel above the orifice EF.

549. If the cylinders *abcd*, ABCD (*fig.* 243, *N.* 2), communicate with each other only by an aperture *ef* in the plane AB, and we abstract

abstract from any pressure upwards upon the lower side of the plane AB, the motion of the water may be determined as in art. 547. The action on the plane CD corresponding to the force that generates the velocity X—V at the aperture EF will be expressed as before by $XX \times \frac{BB-OO}{2B}$. If the aperture *ef* be represented by *o*, and the velocity in *ef* by Y, the action on the surface *cd* corresponding to that which generates the velocity Y—Z in the water issuing at *ef*, will be found as above (by substituting *ef* or *o* for AB) to be $XX \times \frac{CC-oo}{2C} \times \frac{OO}{oo}$, which being diminished in the ratio of CD to *ab* or of B to C, gives $XX \times \frac{CC-oo}{2CC} \times \frac{OO}{oo} \times B$ for the part of this action that is to be ascribed to the gravity of the column LCDM; and the sum of these being supposed equal to $Bb \times \overline{g-f} + Bc \times \overline{g-p}$, we shall have XX to $2gH-2FK$, as 1 is to $1 + \frac{OO}{oo} - \frac{OO}{BB} - \frac{OO}{CC}$; and the descent by which the utmost velocity of the water at the orifice EF would be acquired, is to H in the same ratio; from which it follows (because F is measured by $\frac{\dot{x}}{\dot{t}}$), that this ratio being represented by that of 1 to *m*, the fluxion of the time in which the water issuing at EF acquires the velocity X, will be represented by $\frac{2K}{m} \times \frac{AA\dot{x}}{AA-XX}$, and that this time may be determined by a construction similar to that in art. 538, when the vessel is supposed to be kept always full to the altitude LC. If O be very small compared with B and C, then 1 is to *m* as *oo* to $OO+oo$. And when *ab* is equal to AB, if no water be supplied into the vessel, the velocity is determined by the construction in art. 540, by supposing *e* to represent $\frac{BB-OO}{OO} + \frac{BB-oo}{oo}$.

550. When the vessel consists of any number of cylindric or prismatic parts that have the areas B, C, D, &c. (*fig.* 243) for their several bases, and *b, c, d, &c.* for their respective altitudes, then, by proceeding as in art. 547, the forces that act at the respective surfaces

surfaces B, C, D, *&c.* corresponding to those that are supposed in this method to generate the increase of the motion of the water at each surface will be measured by $XX \times \frac{BB-OO}{2B} \times \frac{OO}{OO}$, $XX \times \frac{CC-BB}{2C} \times \frac{OO}{BB}$, $XX \times \frac{DD-CC}{2D} \times \frac{OO}{CC}$, *&c.* The parts of these forces, which are to be ascribed to the gravity of the column which insists on the lowermost base B, are expressed by $XX \times \frac{BB-OO}{2B} \times \frac{OOB}{OO}$, $XX \times \frac{CC-BB}{2CC} \times \frac{OOB}{BB}$, $XX \times \frac{DD-CC}{2DD} \times \frac{OOB}{CC}$, *&c.* the sum of which is $XX \times \overline{\frac{B}{2} - \frac{OOB}{2SS}}$ if S be the uppermost section of the vessel. But supposing F, *f*, p, &c. to represent the forces described in art. 547, the same sum is equal to $Bb \times \overline{g-f} + Bc \times \overline{g-p} +$ *&c.* or (supposing K equal to $b \times \frac{O}{B} + c \times \frac{O}{C} + d \times \frac{O}{D}$, *&c.*) to BH*g*—BKF. From which it follows that $XX \times \frac{SS-OO}{2SS}$ is equal to H*g*—KF; and that if A represent the velocity which would be acquired by a descent equal to $\frac{SSH}{SS-OO}$, then XX will be to AA as H*g*—KF to H*g*; so that if the water be always supplied at the surface S, with the same velocity with which it subsides at S, when F is supposed to vanish, or the water at EF to have acquired its utmost velocity, X is equal to A. The fluxion of the time is expressed by $\frac{K}{gH} \times \frac{AA\dot{x}}{AA-XX}$ where $\dot{x}$ represents the fluxion of X; and consequently the time is determined as in art, 538, by hyperbolic areas or logarithms. When no water is supposed to be supplied into the vessel, let D be the descent by which X the velocity of the water at EF would be acquired, $\dot{d}$ its fluxion, —$\dot{h}$ the fluxion of H the altitude of the water in the vessel above the orifice, then XX being equal to 2*g*D (art. 434), or $X\dot{x}$ to $g\dot{d}$, the velocity with which the surface of the water subsides, or $X \times \frac{O}{S}$ being expressed by $\frac{-\dot{h}}{\dot{t}}$,

F being

F being expressed by $\frac{x}{t}$ or $\frac{gOD}{hS}$, and $XX \times \frac{SS-OO}{2SS}$ equal to Hg—KF, by what has been shown, it follows that d, the fluxion of D, is to $-h$ the fluxion of H as $H - D \times \frac{SS-OO}{SS}$ to $K \times \frac{O}{S}$, where S always denotes the area of the uppermost surface of the water, O the area of the orifice, H the height of the water in the vessel above O, D the descent by which the velocity X would be acquired, and K is supposed equal to the sum of the products when the altitude of each part of the vessel that contains water is multiplied by the ratio of the orifice to the area of the section of that part. It easily appears that the same conclusions take place when an erect vessel is terminated by any curvilineal surface, supposing K to represent the area of a figure, whose ordinate at any point of the axis is to 1 as the area of the orifice is to the section of the vessel at that point: and these agree with what is deduced by the learned author above mentioned, from the principle described in art. 525 and 532. When any sections of the vessel increase from any part downwards towards the orifice, this theory supposes that there is an action of the water from below upwards, while it passes from narrower into larger parts of the vessel; and in this case the motion of the water does not seem to be so justly determined by it; see art. 527. Several other observations might be made on this doctrine, but our design obliges us to proceed now to other subjects.

551. There are several other principles that relate to the centre of gravity of bodies, besides these we have insisted on hitherto, that are also of use in the resolution of problems. When two powers sustain any body or figure that is supposed to gravitate, a right line from its centre of gravity perpendicular to the horizon passes through the intersection of the right lines in which these powers act, which with the gravity of the figure are in the same ratio to one another as any three right lines constituting a triangle that are parallel to the respective directions of these powers. Hence the nature of the figure is discovered, which is assumed by a heavy chain or perfectly flexible

flexible line that is suspended from any two of its points. Let FEH (*fig.* 244) be such a line, F its lowermost point, where the tangent FT is parallel to the horizon, ED an ordinate from E to the horizontal line AD, ET the tangent at E intersecting FT in T, and G the centre of gravity of the portion FE of the line or chain. Then the three powers are, the gravity of the chain which acts in the perpendicular to the horizon, and the powers at F and E which retain those extremities of the chain, by acting in the tangents FT and ET, and are equal to the tension of the chain at those points. Therefore by this principle the perpendicular from G to the horizon passes through T; and if EI parallel to AD or FT meet TG in I, the weight of the part of the chain FE will be to the tension of the chain at F as IT to EI, or (by prop. 14) as the fluxion of the ordinate DE to the fluxion of the base AD; consequently the tangent of the angle IET, in which the curve intersects a parallel to the horizon at any point E, is always as the weight of the portion FE of the chain that is betwixt E and the lowermost point F; the tension of the chain at any point E, is to its tension at F as ET to EI (by the same principles) or as the fluxion of the curve to the fluxion of the base, and is as the secant of the angle IET. We shall afterwards consider this subject in a more general manner. When any body or number of bodies connected together are suspended in any manner, their common centre of gravity descends to as low a place as possible; and hence some problems have been resolved concerning the *maxima* and *minima*; but of these we are to treat afterwards, and proceed now to some general observations on the subjects of the 10th and 11th chapters, whence we shall endeavour to draw some general principles that may be of use in resolving philosophical problems of various kinds.

552. It was observed above in art. 312, that the asymptote of the branch of a curve is considered as the tangent at its infinitely distant extremity. In prop. 26, while P describes the branch that approaches to the asymptote RX (*fig.* 117), let CP and SP meet RX in m and n; and when the revolving lines CP and SP become parallel to one another and to RX, their angular velocities will be in the ultimate ratio of the angles PCx and

PSy,

PSy, or of CmR, and SnR, and consequently in the ratio of CR to SR; so that SQ will be to CQ as CR is to SR, and CR equal to SQ. And thus the demonstration of the 26th proposition may be abridged, the use of which has been shown by many examples in chap. x.

553. The propositions in chap. xi. concerning the curvature of lines and its variation may be likewise briefly demonstrated from the limits of ratios. Let TR (*fig.* 149) parallel to EB meet the curve EMH in M, the circle ERB in R, and their common tangent in T, as in prop. 32; then supposing ET to be continually diminished till it vanish, the ultimate ratio of TM to TR will be the ratio of the curvature of the line EM at E to the uniform curvature of the circle ERB; and the rays of curvature will be in the inverse ratio. When this is a ratio of equality, no circle can pass between EM and ER within the angle of contact REM, and ERB is the circle of curvature at E. Because TM, ET, and TK are supposed to be in continued propropotion (art. 366), and when ET represents the fluxion of the curve, TM ultimately measures one half of the second fluxion of the ordinate, and TK ultimately coincides with EB; it follows that the right lines which measure the second fluxion of the ordinate and the first fluxion of the curve and $\frac{1}{2}$ EB are in continued proportion, as was shown at greater length in prop. 33. When we speak of the ratio of a fluxion to a fluent, we always understand the ratio of the right lines that represent them.

554. Angles of contact are in the ultimate ratio of their subtenses, when the arches, or their tangents, are supposed to be equal, and to be continually diminished till they vanish, if the subtenses are inclined in equal angles to those tangents. It was shown in art. 369, that RM the subtense of the angle of contact MER contained by the curve EM and circle of curvature ER was as KQ directly, and the rectangle KTQ inversely, ET being given. Therefore when EB is the diameter of the circle of curvature, and BV the tangent of BK is not parallel to ET, the angle of contact MER is as the tangent of the angle BVE directly, and the square of the ray of curvature inversely; and when the curvature at E is given, the index of the variation of curvature (according to Sir *Isaac Newton*'s explication)

plication) is as the angle MER (*fig.* 152). When the curve BK touches the circle BQ at B, if C and O be the respective centres of curvature of BQ and BK at B, then KQ is as OC directly, and the rectangle OBC inversely, and the angle of contact MER is as OB directly, and CB^4 inversely; and when the arches EM and E*m* are similar in this case, the angle MER is to *mer* in the triplicate ratio of E*b* to EB. The angle of contact, for example, contained by the parabola and the circle of curvature at its vertex is inversely as the cube of the parameter of the axis. When the contact of BK and BQ is of any order denoted by *n*, according to the explication in art. 369, then the angle MER in similar arches is inversely as the power of the ray of curvature the exponent of which is $n+2$.

555. The rest remaining, let MN (*fig.* 245) perpendicular to the tangent at M, and M*d* perpendicular to the chord EM meet the ray of curvature FC in N and *d* respectively; then the last ratio of EN to the ray of curvature EC and of E*d* to 2EC will be a ratio of equality. For E*d* is to TK as EM^2 to ET^2, and the excess of E*d* above TK to TK as TM^2 to ET^2 or MTK, that is, as TM to TK; consequently E*d* always exceeds TK by TM, which excess vanishes with ET when TK coincides with EB. The fluxion of E*d* is equal to the fluxion of TK when M sets out from E, and may serve for measuring the variation of curvature at E, by art. 369 and 386.

556. Any arch being given, the centre of its curvature is the limit of the intersections of right lines that bisect perpendicularly the sides of the rectilineal inscribed or circumscribed figures when the arch (with those figures) is continually diminished till it vanish; and is also the limit of the intersections of right lines that bisect the angles of those figures. But the intersection of right lines perpendicular to those sides at their extremities will not coincide ultimately with the centre of curvature (*fig.* 246). Let *ux* bisect any chord M*m* perpendicularly in *u*, and meet the ray of curvature EC in *r*, then C will be the limit of all the situations of the point *r* when the arch EM*m* is supposed to be diminished till it vanish; but if *ms* perpendicular to M*m* at *m* meet EC in S, the ultimate ratio of ES to EC will be the same with the ultimate ratio of E*m* to EM + ½ M*m*; so that if

E*m*

Em be to EM as m to n, the ultimate ratio of ES to EC will be that of $2m$ to $2m-n$.

557. Supposing as above ET (*fig.* 245) to be the tangent of EM at E, TM the subtense of the angle of contact parallel to EB, TK to be always equal to $\frac{ET^2}{TM}$, and FK the *locus* of the point K to intersect EB in B; it is manifest that when ET is supposed to be continually diminished and at length to vanish, TK then coincides with EB; and this seems to be sufficient to justify the expression, when it is said that EB is the ultimate value of $\frac{ET^2}{TM}$ which is supposed to be always equal to TK. But if it should be objected, that when ET vanishes TM likewise vanishes, the ratio of ET to TM is not assignable, and the value of $\frac{ET^2}{TM}$ must therefore be then inconceivable or imaginary. In answer to this we may observe first, that nothing is more usual in Geometry than to determine the points of one figure from those in another by a construction or equation, as in this case any point K in FKB from the corresponding point M in HME by supposing TK always equal to $\frac{ET^2}{TM}$; that the point in the former which corresponds to E in the latter can be no other than B where the *locus* FKB intersects EB; that EB must either be allowed to be the ultimate value of $\frac{ET^2}{TM}$, or we must only say that $\frac{ET^2}{TM}$ is equal to TK with the single exception of the case when T falls on E: and as it has not been usual, or thought necessary, to require so scrupulous an exactness, so it seems unreasonable to find fault with the inventor of this method for making use of a convenient and concise expression that is not liable to more exceptions than such as were allowed before his time. When EMH is an arch of a semicircle described upon the diameter EB, FKB is an arch of the same semicircle, and $\frac{ET^2}{TM}$ is generally allowed to be always equal to TK or EB — TM without excepting any particular case; from which it

it would follow that since EB — TM becomes equal to EB when ET and TM vanish, the ultimate value of $\frac{ET^2}{TM}$ is EB. But there is no necessity for making use of exceptionable expressions in any part of Geometry; and the same author has shown us how to avoid them in this case. For we may consider EB as the ultimate value of TK, but only as the limit of the values of $\frac{ET^2}{TM}$ when ET is continually diminished till it vanish; and such a limit may be understood to be always meant by what is called the ultimate value of a quantity that is determined in this manner from others that vanish together. There can be no flexure or curvature in a point, and the curvature at E has indeed a dependence on the values of $\frac{ET^2}{TM}$ when ET and TM are real, but in so far only as the value of their limit EB has a dependence on those values; for it was shown in prop. 32, that in order to determine the curvature at E (as it was defined in art. 364), it is sufficient to ascertain the distance EB. This is no more than one of those problems that frequently occur, the determining the intersection of a curve with a right line given in position; and it is, generally speaking, more easy to determine the point B than the intersection of FKB with any other parallel TK.

558. When S (*fig.* 245) is any given point in EB, let SM meet the tangent ET in l, and lM will be to TM as Sl to SE, which is ultimately a ratio of equality; consequently the ultimate value of $\frac{ET^2}{lM}$ is the same as of $\frac{ET^2}{TM}$ and is equal to EB; the same is to be said of $\frac{EM^2}{lM}$ or $\frac{EM^2}{TM}$.

559. The tangents of the evoluta aCI (*fig.* 180), intercepted by AEM give a convenient scale of the rays of curvature of the latter. And if these rays CE, QM be divided in Z, z, so that CZ be always to EZ in the same given ratio of m to n, and the tangent of the *locus* of Z meet ET perpendicular to CE in t, the variation of curvature at E will be always as $\frac{m}{n}$ × tangent Et Z.

For

For let an arch Zx described from the centre C meet QM in x, and the last ratio of EM to Zx will be that of EC to ZC; and because Zx is ultimately equal to $CQ + CZ - Qz$, and $Qz - CZ$ is to $QM - CE$ (or CQ) as CZ to CE, the last ratio of zx to CQ is that of EZ to EC. Therefore the last ratio of xz to Zx is that of $CQ \times EZ$ to $EM \times ZC$, or of $n \times CQ$ to $m \times EM$; consequently the last ratio of CQ to EM, or of the fluxion of the ray of curvature CE to the fluxion of the curve AE (which ratio measures the variation of curvature), is that of $m \times xz$ to $n \times Zx$, or of $m \times EZ$ to $n \times Et$, or of $\frac{m}{n} \times$ tang. EtZ to the radius. It is easy to show, from art. 384, that in all figures wherein the sine of the angle contained by the ordinate and curve is as a power of the ordinate whose exponent is any number r (as for example in the cycloid, catenaria, elastic curve, &c.), the ray of curvature EC always meets the base at Z so that EZ is to EC in the invariable ratio of 1 to r; consequently the base being the *locus* of Z, the variation of curvature in such figures is as $\frac{r-1}{r} \times$ co-tang. of the angle contained by the ordinate and curve.

560. When EMH (*fig.* 247) is described by a gravity that acts at E in the direction EB, let EK be the space that would be described by a body falling from E in the right line EB by the gravity at E continued uniformly in the same time that the tangent ET would be described by the motion in the trajectory at E; then this time being given, the gravity at E will be measured by 2EK, because a force is measured by the motion which it would generate in a given time, and a space 2EK would be described by the motion acquired at K in the time that EK would be decribed by the body descending from E to K, by art. 95. But when ET is continually diminished till it vanish, the ultimate ratio of TM to EK is a ratio of equality; and the velocity in the trajectory being measured by ET, the gravity at E will be in the ultimate ratio of 2TM. It is usual in enquiries of this nature first to consider the motion as uniform in the chords mE, EM inscribed in the figure, or in its tangents, and to conceive the gravity to be applied at once at the angle E. Let RM pa-

rallel to EB meet the chord mE produced in R and the tangent at E in T, then the ultimate ratio of RM to 2TM will be a ratio of equality, and the gravity at E will be in the ultimate ratio of RM or 2TM, whether it be conceived to act at once at E (as in *prop.* 30, *lib.* 3, *Princip. Edit.* 3), or to act continually, the velocity at E being in the ultimate ratio of ET or EM. Let EM the side of the inscribed figure be bisected in L, and the angle ELd being supposed equal to MTE, let Ld meet EB in d, and the triangles MTE, ELd being similar, Ed will ultimately coincide with Eb half the chord of curvature EB; and the ultimate ratio of the rectangle RM $\times$ Eb to EM^2 will be a ratio of equality; or the rectangle contained by half the chord of curvature and the right line which measures the gravity, equal to the square of that which measures the velocity at E, as in art. 464.

561. In like manner if we suppose mEM to be any arch of a perfectly flexible line or chain, n to denote the section of that chain at E perpendicular to its length, EK the accelerating force of the gravity at E, then EK $\times$ n $\times$ EM will express the absolute gravity of an uniform chain equal in length to EM of a base equal to n that is acted upon by the force EK; and this is ultimately equal to the absolute gravity of the portion EM of the chain; consequently the tension of mEM at E is measured by the ultimate value of EK $\times$ n $\times$ EM $\times$ $\frac{EM}{RM}$ or of EK $\times$ n $\times$ Eb, and is equal to the weight of a chain equal in length to Eb of the same thickness with AEB at E that is acted upon by an uniform gravity equal to EK.

562. Let E (*fig.* 248) by any point in IL a right line given in position, A a given point that is not in IL, join AE, and let AC perpendicular to AE meet IL in C. Then if we suppose the point E to move in IL, but C to remain, AE and CE will flow proportionally; that is, the fluxion of AE will be to the fluxion of CE as AE to CE. For let AK be perpendicular to IL in K, and the fluxion of AE will be to the fluxion of KE as KE to AE (by prop. 15), or as AE to CE; and the fluxion of CE is equal to the fluxion of KE when C is supposed to remain fixed. When the point A is taken any where upon an arch described from the

centre

centre C, and AE the tangent of this arch at A meets the diameter IL given in position in E, then the point A being supposed to remain if E move in the right line CE, the fluxion of AE will be always to the fluxion of CE as AE to CE. The converse of which is, that when the fluxion of AE is always to the fluxion of CE as AE is to CE, the point A being taken any where on the circular arch, and E being supposed to move in CE, then C is the centre of the arch. In general let the fluxion of AE be always to the fluxion of cE as AE is to cE, the points A and c being supposed to remain; and if while the point A approaches to the right line IL till it coincide with it, the point c approach to C as the limit of its various positions, then is C the centre of the curvature of the line upon which A is supposed to move at that point of it where A falls upon IL.

563. These observations lead us to some general propositions relating to philosophical enquiries, which we shall represent in one view, that the analogy which is between them may the better appear. The first gives the property of the trajectories that are described by any centripetal forces how variable soever these forces or their directions may be: the second gives a like general property of the lines of swiftest descent: the third gives the property of the lines that are described in less time than any other of an equal perimeter: and the fourth gives the property of the figure that is assumed by a flexible line or chain in consequence of any such forces acting upon it. Let AEB (*fig.* 249) be an arch of any of those lines, HEh a right line in the direction of the power EK that results from the composition of the several forces that are supposed to act at E, and let a perpendicular from O, the centre of curvature at E, meet HE in C.

I. The velocity in the trajectory at E is equal to that which would be acquired by a descent equal to $\frac{1}{4}$ CE by an uniform gravity equal to EK the force which acts at E. And if we suppose a body to set out from E in the right line HEh with a velocity equal to that in the trajectory at E, and its motion to be accelerated or retarded by the same powers that act at E, then its velocity and distance from C will increase proportionally; that is, the fluxion of the right line V, which represents its velocity, will be to the fluxion of its distance from C as V is to

 the

the distance CE. Or, in other words, if EN the ordinate of the figure HNG measure its velocity at any point as E of Hh, and NT the tangent of HNG at N meet Hh in T, the subtangent ET will be equal to EC on the opposite side of E.

II. The velocity in the line of swiftest descent AEB at E is equal to that which would be acquired by an uniform gravity equal to EK, the force that acts at E, by a descent equal to $\frac{1}{2}$ CE. The curvature of this line at E is equal to the curvature of the trajectory that would be described by a body projected from E in the direction of the tangent of AEB with the velocity acquired in AEB at E, and that is acted upon by the same force EK. And in this case likewise V and CE flow proportionally; or ET the subtangent of the figure HNG and EC half the chord of curvature coincide with one another.

III. When the sum or difference of the time in which the line AEB is described, and of the time in which it would be described by an uniform motion with a given velocity is a *minimum*, the line AEB will then be described in less time than any line of an equal perimeter that has the same extremities A and B. And it is a property of such lines that if a body set out from E with the velocity u acquired at E in EH or Eh, the fluxion of u will be to the fluxion of its distance from C in the compound ratio of u^2 to CE, and of the sum or difference of b and u to a, b and a being supposed to represent invariable velocities. By principles analogous to this, the nature of the line that is described in less time than any line that includes the same area AEB with the chord AB in any hypothesis of gravity may be discovered, and other problems of this kind concerning isoperimetrical figures resolved.

IV. When AEB is a flexible line or chain, its tension at E is equal to the weight of a chain that is in length equal to CE, of an uniform thickness equal to that of AEB at E, and that is acted upon by an uniform gravity equal to EK the force that results from the composition of the several powers that act at E. Let A be a given point in the chain AEB, Aa equal to one half of the chord of the circle of curvature at A, that is in the direction of the force which acts on the chain at A. Let Ek be always to EK the force that acts at any point E as the section of the chain

chain at E to its section at A, and the direction of the force E*k* be opposite to that of EK; then if a body set out from A with a just velocity (*viz.* that which would be acquired by a descent equal to *a*A, by an uniform gravity equal to the force that acts at A), and while it is made to move along the curve AEB, its motion be always accelerated or retarded by the forces represented by E*k*, the tension of the chain at any point E will be always in the duplicate ratio of the velocity acquired at E; which is the same velocity that would be acquired by the descent CE with an uniform gravity equal to the force E*k*. And if abody be projected from E with this velocity in the direction of the tangent of AEB, the curvature at E of the trajectory that would be described by the force E*k* will be one half of the curvature of the chain at E.

564. The first of these follows easily from what was shown above in art. 464 or 560. For the fluxion of the velocity EN being in the compound ratio of the force EK and of the fluxion of the time, which is as the fluxion of the distance CE directly (the point C being supposed to remain fixed), and the velocity inversely, it follows that the fluxion of EN is to the fluxion of CE as EK is to EN, or as EN to EC; but the fluxion of EN is to the fluxion of CE as EN is to ET; consequently CE is equal to ET. But having insisted at length on this subject in the last chapter, we have mentioned this theorem here for the sake of its analogy to the rest only.

565. Let A and B (*fig.* 250) be two given points, IL a right line that bisects AB perpendicularly in K; and it is manifest, that if a body is to move from A to B in the least time with a given uniform motion, it must describe the right line AB; and if it is to move from A to the right line IL in the shortest time, it must describe the perpendicular AK. But E being any point upon IL, join AE and BE; and if we now suppose that the body is to describe AEB with an uniform motion, but with a velocity that is always as CE, the distance of E from C a point given upon IL, then the motion will not be performed in the least time when E falls upon K, but when AE is perpendicular to AC. For let KR parallel to AE meet AC in R, and the time in which any line AE is

described will be always directly as AE, and inversely as the velocity or CE; that is, the time will be as KR, since KR is to KC as AE to EC, and KC is given; but KR is least when it is perpendicular to AC; consequently AE is described in the least time when AE is perpendicular to AC. It follows, conversely, that if AE or AEB be described in the least time, and the velocity be as the distance of E from some point upon IL, that point must be C, where AC perpendicular to AE intersects IL. And this with art. 562, suggests the general property of the curvature of the lines of swiftest descent, that if IL meet this line in E, and the velocity in IL be as the distance from C, or, more generally, if (the point C remaining) when CE increases or decreases the velocity at E begin to flow in the same proportion as CE, then the flexure of the line of swiftest descent at E must be such as to have the centre of its curvature in C. In this investigation of the curvature of the line of swiftest descent, we conceive AE and EB not to be the whole chords that form the rectilineal figure inscribed in it (or the whole tangents that form the circumscribed figures), but their halves only, and any two such successive parts to be described uniformly with the velocity pertaining to their intersection E, which is ultimately the mean velocity in the arch, and the centre of curvature to be determined by the ultimate intersection of the perpendiculars AC, BC with each other, or with IL that bisects the angle AEB, according to art. 562. But the nature of the line of swiftest descent may be discovered more easily than from this property, when the gravity acts in parallel lines, or is directed towards a given centre; and that this theory may be set in a clear light, we shall treat of it and the higher problems concerning the *maxima* and *minima* in a separate chapter.

566. The first part of the fourth theorem, that was proposed in art. 563, has been already demonstrated in art. 561, *viz.* that the tension of the line or chain AEB (*fig.* 249), at any point E, is equal to the weight of a chain of the same thickness with AEB at E that is in length equal to EC and is acted upon by an uniform gravity equal to EK, and consequently is measured by the rectangle kEC. As to the latter part, let kr be perpendicular to the tangent of AEB at E in r; let V be a right line determined

determined from the forces Ek, as in art 435, so as to represent the velocity which the body is supposed to acquire at E, while it moves along AEB in the manner described in the theorem; then because Er is the force by which that velocity is accelerated or retarded, the rectangle contained by Er and the fluxion of the curve AE will measure the fluxion of $\frac{1}{2}$VV. But because Ek is in the compound ratio of the force EK and thickness of the chain at E, Er is the force by which the tension of the chain increases from the point E, and the rectangle contained by Er and the fluxion of the curve AE will measure the fluxion of the tension at E or of the rectangle kEC. Therefore since $\frac{1}{2}$VV is supposed equal to the rectangle kEC when the point E falls upon A, they will be always equal to each other. Let Ek meet in Q the circle of the same curvature at E with the trajectory described by the force Ek, when the body is projected from E with the velocity V in the direction Er; and by what has been shown above, if EQ be bisected in b, the rectangle bEk will be equal to VV, and consequently to 2CE$\times$Ek. Therefore Eb is equal to 2EC, or the curvature of the trajectory at E is one half of the curvature of the chain at E.

567. When EK (*fig.* 251) is either a centripetal or centrifugal force that is directed towards a given point S, or from it, take SM upon SA always equal to SE, let the ordinate MN of the figure adNM be always equal to Ek, and if the area aAgd be equal to $\frac{1}{2}$VV when the body sets out from A, or measure the tension at A, the area adNM will always measure $\frac{1}{2}$VV or the tension at any point E. And if SP be perpendicular to the tangent of the *catenaria* AEB at E, this perpendicular SP will be always inversely as the area adNM, or inversely as the tension at E, or inversely as VV the square of the velocity acquired at E. For, since the fluxion of SE is to the fluxion of the curve AE as Er is to Ek, it follows that the fluxion of $\frac{1}{2}$VV is equal to the rectangle contained by Ek and the fluxion of SE; so that the fluxion of V is to the fluxion of SE as Ek is to V or $\frac{1}{2}$V to EC. But the fluxion of SE is to the fluxion of SP as EC is to SP, by art. 384, consequently, the fluxion of V is to the fluxion of SP as V to SP; and since SP decreases while V increases, it fol-

lows that SP is inversely as VV or $adNM$. Hence an analogy appears between these figures and the trajectories described by centripetal or centrifugal forces; in these, SP the perpendicular from the centre S upon the tangent of the trajectory is inversely as the velocity of the body that describes it; whereas in those, SP is inversely as the square of V the velocity of the body that moves along the curve, when the direction of the force is changed according to the fourth theorem in art. 563. If the force EK be invariable, for example, and the chain is of an uniform thickness, and if Sa vanish (that is, if the tension at A be equal to the weight of a chain of the same thickness with AEB at A, equal in length to SA, and that is acted upon by an uniform gravity equal to the force EK), SP is inversely as SE, and consequently AEB is an equilateral hyperbola, as Mr. *Herman* observes. But AEB is not always such an hyperbola, when the force towards S is uniform, as this learned author seemed to think. For when the tension at A is different, SP is inversely as aM or SE + Sa. In like manner when the chain is of an uniform thickness, and EK is a centrifugal force that is inversely as the square of the distance from S, and the tension at A is equal to the weight of a chain of the same thickness equal in length to SA and that is acted upon by an uniform gravity equal to the force at A, AEB is an arch of a logarithmic spiral. When the force EK is centrifugal and inversely as the cube of the distance from S, AEB in a similar case is an arch of a semicircle described upon the diameter SA; and when EK is as some other power of the distance, AEB is in similar cases one of those figures that were constructed in art. 392 or 393.

568. When the force EK (*fig.* 252) acts in parallel lines, the sine of the angle DEP contained by the curve and the ordinate that is in the direction of the force is inversely as VV, or the area adNM, or the tension at E. If the force Ek be uniform, let ED in the direction of the forces meet ad in D, and the sine of DEP being inversely as aM or DE, DP perpendicular to EP the tangent at E will be invariable. In this figure the rectangle kEC is equal to aM × Ek, and EC equal to aM or DE; and the ray of curvature EO being to EC (or DE) as DE to DP, EO is inversely as the square of the sine of the angle DEP.

569. When

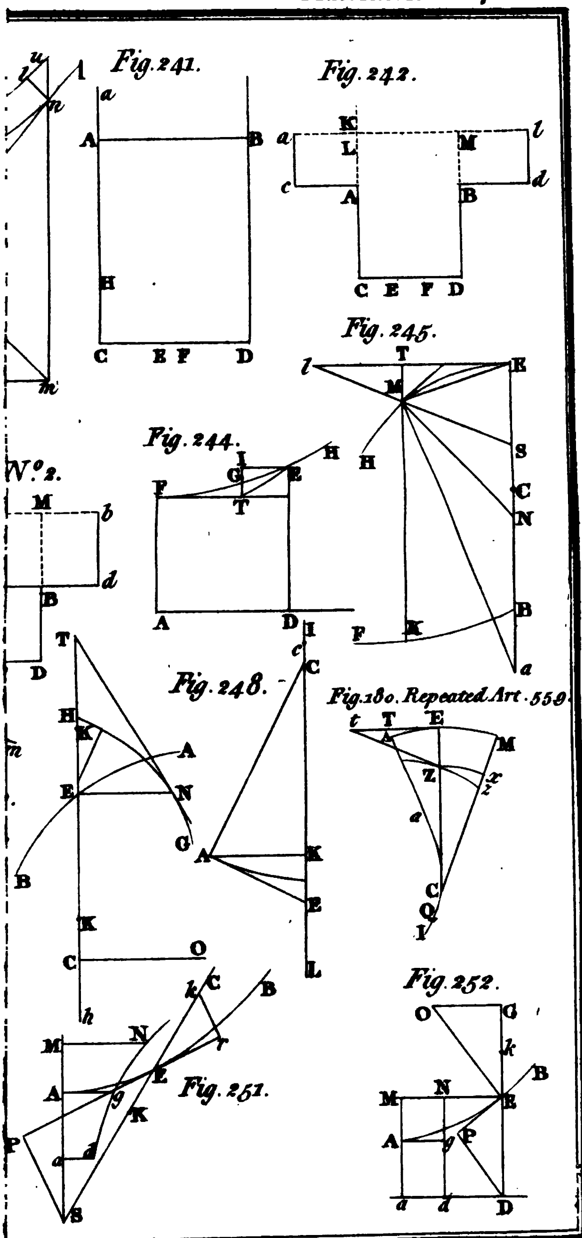
Fig. 241.
Fig. 242.
Fig. 245.
Fig. 244.
N°. 2.
Fig. 248.
Fig. 180. Repeated Art. 559.
Fig. 252.
Fig. 251.

569. When the force EK is perpendicular to the curve, it has no effect on V, so that V is in this case constant, and the tension is the same in all parts of the figure, and EC (which in this case is the ray of curvature) is inversely as the force that acts at E. This force in the *velaria* (according to Mr. *Bernouilli*) is as the square of the sine of the angle in which the ordinate intersects the curve; so that the ray of curvature must be inversely as that square, and the *velaria* must coincide with the common *catenaria*, by what was observed at the end of the last article. In the elastic curve the force EK is as the ordinate, and the ray of curvature is inversely as the ordinate.

570. When a power EI (*fig.* 253) always perpendicular to the curve, and a centripetal or centrifugal force EL always directed towards or from a given centre S, act at once upon a line or chain AEB of an uniform thickness, the former has no effect upon V; and if LZ be perpendicular to the ray of curvature in Z, and SM being equal to SE, the ordinate MN be always equal to EL, as in art. 567, the rectangle contained by the ray of curvature EO and $EI \mp EZ$ will be always equal to the area *ad*NM. For complete the parallelogram ELKI, let KR be perpendicular to OE in R, and the rectangle REO will be equal to the rectangle KECor ½VV which is equal to *ad*NM; and since IR is equal to EZ, it follows, that *ad*NM is equal to the rectangle contained by EO and the sum or difference of EI and EZ. It is manifest that EZ is to EL the force that acts in the right line ES, as SP the perpendicular from S on the tangent at E is to SE; and that when EL acts in parallel lines, EZ is to EL as the fluxion of the base is to the fluxion of the curve; in which last case this theorem agrees with what is shown *comment. petropol. tom.* 3. The property of the ray of curvature being thus discovered, the nature of the figure may in some cases be defined by first fluxions, or by a common equation, by a proper application of the inverse method of fluxions. The problems in art. 563, considered in a general manner, depend on the curvature of lines; and therefore the general solution involves the ray of curvature, or something equivalent. But there are often particular principles which serve for resolving more easily particular cases of those problems, of which

which we gave instances in art. 441 and 551 (where the solution agrees with that in art. 568), and we shall have occasion to give other instances in the following chapter relating to the lines of swiftest descent.

CHAP. XIII.

Wherein the Nature of the Lines of swiftest Descent is determined in any Hypothesis of Gravity, and the Problems concerning isoperimetrical Figures, with others of the same Kind, are resolved by first Fluxions, and the Solutions verified by synthetic Demonstrations.

571. IT was shown in chap. ix. how the greatest and least ordinates of figures are readily determined by the method of fluxions, where the usual rules with the corrections that are necessary to render them accurate and general were demonstrated. But there are problems concerning the *maxima* and *minima* which are of a higher nature, that cannot be immediately reduced to these. It was known long ago that of all equal areas the circle has the least circumference, and of all equal solids the sphere is bounded by the least surface. But the first problem of this kind that required a more subtile investigation, seems to have been resolved by Sir *Isaac Newton, Schol. prop.* 34, *lib.* 2, *Princip.* where he gives the property of the figure, that by revolving on its axis generates the solid of least resistance. Afterwards Mr. *Bernouilli* found, that the cycloid was the line of swiftest descent in the common hypothesis of gravity, and determined the nature of this line in several other cases and under various restrictions. The analysis of the general problem concerning figures, that amongst all those of the same perimeter produce *maxima* and *minima* was given by Mr. *James Bernouilli*, from computations that involve second and third fluxions, by resolving the element of the curve into three infinitely small parts.* And several enquiries of this nature have

* *Analysis magni problematis isoperimetrici.* Acta erud. Lips. 1701, *p.* 213, *& seqq*

have been since prosecuted in like manner, but not always with equal success. In pursuit of our principal design in this treatise, of vindicating this doctrine from the imputation of uncertainty or obscurity, we shall endeavour to illustrate this subject, which is commonly considered as one of its most abstruse parts, by proposing the resolution and composition of these problems, and to determine the properties of the lines of swiftest descent (whether gravity be supposed to act in parallel lines, or to be directed to a given centre, and whether the perimeter of the figure be supposed to be a determinate quantity, or other limitations of this kind be added or not), and of the isoperimetrical figures that produce other *maxima* and *minima* immediately by first fluxions, without resolving the elements of the curve into two or more parts, and in such a manner as may suggest a synthetic demonstration that may serve to verify the solution. The whole might be contained in a few general propositions; but it may be useful in this, as in the preceding chapters, to begin with the more simple cases, and to proceed from them to such as are more complex. We shall therefore first suppose the gravity to act in parallel lines.

572. The following *lemma* is to be premised. Let KL (*fig.* 254) be a right line given in position, AK a perpendicular upon KL from a given point A, E any point in this line, join AE, suppose KE to be described uniformly with any given velocity a, and AE to be described uniformly with any given velocity u that is less than a: let L be taken upon the right line KL, so that AL may be to KL as a is to u; and the difference of the times in which the right lines AE and KE will be described by the respective velocities u and a (or $\frac{AE}{u}-\frac{KE}{a}$) will be least when E falls upon L; that is, when the angle KAE is such, that its sine is to the radius as u is to a. For let KH and EV be perpendicular on AL in H and V respectively, and AR be taken upon AL equal to AE; then HV will be to KE as KL is to AL, or (by the construction) as u to a; consequently HV will be described with the velocity u, in the same time that KE is described with the velocity a. Therefore the excess of the time in which AE is described with the velocity u, above

above the time in which KE is described with the velocity a, is equal to the time in which AE——HV, or AR——HV, or AH + VR is described with the velocity u; and because AH is invariable, this time is least when VR vanishes; that is, when E falls upon L, and the sine of the angle KAE is to the radius as u is to a. The same appears from art. 242, according to which $\frac{AE}{u}-\frac{KE}{a}$ is a *minimum* when its fluxion vanishes, that is (because u and a are supposed to be invariable), when u is to a as the fluxion of AE to the fluxion of KE, or (by art. 193) as KE is to AE, that is, when E falls upon L.

573. It follows, that if kl, KL be any two parallel lines, e any point upon kl, and E any point upon KL, the right line eE be described with the velocity u, and eb being perpendicular to KL in b, bE be described with the velocity a, the difference of the times in which eE and bE are thus described will be least when the sine of the angle Eeb is to the radius as u is to a; and that when it is required that this difference should be a *minimum*, the angle Eeb does not depend on the magnitude of eb, but on the ratio of u to a only.

574. The gravity being supposed to act in parallel lines, suppose FED (*fig.* 256) to be the line of swiftest descent from the point F to any given vertical HD. Let AE be any arch of this line (the point E being supposed to be lower than A), KE a parallel through E to the horizontal line FH, and AK perpendicular to KE. Then the excess of the time of descent in the arch AE above the time in which KE would be described uniformly by the motion acquired at D is always a *minimum*, AK being given. Let Ae and Ae be any other lines drawn from A to any points in KE on either side of E; let the time of descent in AE be expressed by T . AE; and in like manner let the times of descent in Ae and Ae, and the times in which KE, Ke, and Ke would be described by the motion acquired at D, be expressed by prefixing T to each; then I say that T . AE—T . KE will be less than T . Ae—T . Ke, or T . Ae—T . Ke. To demonstrate this, we are first to observe, that no point of the line FED betwixt F and D can be lower than D; for let FzD be any line that has a point z betwixt F and D lower than D, and

and let zr parallel to FH meet HD in r, then zr will be described in less time than zD, and Fzr in less time than FzD, so that FzD cannot be the line of swiftest descent from the point F to the vertical HD. This being premised, let e be any point betwixt K and E, and e any point on the other side of E; let ed and ed be lines equal and similar to ED and similarly situated, so that eE may be equal to dD, and Ee to Dd. Then by the supposition the time of descent along AED is less than the time of descent along AedD, and by substracting the equal times of descent along ED and ed, it follows that T . AE is less than T . Ae + T . dD, or T . Ae + T . eE, or T . Ae + T . KE — T . Ke. Therefore T . AE — T . KE is less than T . Ae — T . Ke. Let ed meet HD in any point x, and since D is the lowermost point of FED, d must be the lowermost point of ed, and x must be above D. By the supposition the time of descent along AED is less than the time along Aex, and the time in Dd being less than the time in xd, it follows that the time of descent along AEDd is less than along Aed, and by subducting the equal times along ED and ed, it follows that T . AE + T . Dd is less than T . Ae; that is, T . AE + T . Ee, or T . AE + T . Ke — T . KE, is less than T . Ae. Therefore T . AE — KE is less than T . Ae — T . Ke.

575. This property of the line of swiftest descent suggests immediately the nature of the figure. Let AT the tangent of this line at A meet Ke in T, let the velocity acquired at A be called u, and the velocity acquired at D be called a. It is manifest that when AK is continually diminished till it vanish, the ultimate ratio of the time of descent along AE to the time in which AT would be described with the velocity u is a ratio of equality; and that the ultimate ratio of KE to KT, or of the times in which KE and KT would be described with the velocity a, is likewise a ratio of equality. Therefore, since the excess of the time of descent along AE above the time in which KE would be described with the velocity a is always a *minimum*, it follows that the difference of the times in which AT and KT would be described with the respective velocities u and a is a *minimum*, AK being given. Therefore by art. 572, the

the sine of the angle KAT is to the radius as u is to a; and if Aa be the ordinate from A to FH, the fluxion of the base Fa will be always to the fluxion of the curve FA as the velocity at A to the velocity at D. And this is the *analysis* of the problem when the gravity acts in parallel lines. It is obvious, that the line of swiftest descent from F to the vertical line HD is likewise the line of swiftest descent from F to D, or betwixt any two points of FED. Because u becomes equal to a when E comes to D, the curve is therefore perpendicular to HD at D.

576. It will now be easy to show by a synthetic demonstration, that the line which has this property is the line of swiftest descent. Suppose that FAEBD (*fig.* 256) is a line of such a nature that the sine of the angle contained by it at any point E and EQ, the ordinate perpendicular to the horizon, is always as the velocity of the body that descends along it at E; let AEB be an arch of this line, Aa and Bb ordinates perpendicular to the horizontal line FH, AP, and Bp parallel to FH, AT the tangent at A, TEt the tangent at E, tB the tangent at B, and TB, tr parallel to FH. It appears from art. 573, that if AT, Tt, and tB be described uniformly with the respective velocities that are acquired at the points of contact A, E, and B, the excess of the time in $ATtB$ above the time in which ab would be described with any given velocity a greater than that which is acquired at B, will be less than if the points T, t, and B were taken any where else upon the parallels TR, tr, and Bp. Let TR and tr meet the curve in g and h, and Sgf, Vhv the tangents at g and h meet AT, Tt, and tB in S, f, V, and v respectively; and AS, Sf, fV, Vv, vB be described with the respective velocities that are acquired at the respective points of contact A, g, E, h, B, then the excess of the time in which the circumscribed figure $ASfVvB$ is thus described above the time in which ab would be described by the given velocity a, will be less than if the points S, f, V, v, and B were taken any where else upon the right lines SX, fx, VZ, vz, and Bp parallel to FH, by the same article. By increasing in this manner the sides of the circumscribed figure, and supposing each side to be described always with the velocity acquired at its contact with the curve, the time in which the circumscribed figure would be thus described

scribed will approach continually to the time of descent in the arch AEB, and the ultimate ratio of those times will be a ratio of equality; and consequently the excess of the time of descent along AEB, above the time in which *ab* would be described with the given velocity *a* will be a *minimum*, the point A with the distance between the parallels AP and B*p* being given, and the velocities being always the same in the same horizontal lines. Therefore since *ab* is given when the points A and B are given, and the velocity *a* is given, the time in which *ab* would be described with this velocity is given; consequently AEB will be described in less time than any other line A*e*B that passes through A and B. It appears easily that FED perpendicular to HD is the line of swiftest descent from F to HD.

577. When the gravity is uniform, the velocity at E(*fig.* 256, *N.* 2) is in the subduplicate ratio of the ordinate QE; so that (by what has been shown) the fluxion of the base FQ is to the fluxion of FE the line of swiftest descent in the subduplicate ratio of QE to HD; and this being the property of a cycloid that has FH for its base, and HD for its axis, the cycloid is therefore the line of swiftest descent in the common hypothesis of gravity. When the gravity is as the power of the distance from FH whose exponent is any number *n*, and the body is supposed to descend from FH, or to descend from any point A with the velocity that would be acquired by the descent *a*A, then the velocity at E is as the power of QE whose exponent is $\frac{1}{2}n + \frac{1}{2}$; and the fluxion of the base FQ is to the fluxion of FE the line of swiftest descent, as that power of QE is to the same power of HD. In those cases, if EI always perpendicular to the curve meet FH in I, the motion of the point I in the right line FH will be uniform while the body descends along the curve, and may serve to measure the time of descent. The velocity of I in the right line FH will be to the velocity acquired at D, the lowermost point of the line of swiftest descent, as the difference betwixt 1 and *n* is to 2; and the time in which HI would be described uniformly with the motion acquired at D, will be to the time of descent in ED in the same ratio. Let FQ be supposed to flow uniformly, then (by the property of the line of swiftest descent) the fluxion of FE will be inversely as the velocity

locity at E, or the power of EQ whose exponent is $\frac{1}{2}n + \frac{1}{2}$, and (by art. 167) the second fluxion of FE will be to the fluxion of EQ in the compound ratio of $n + 1$ to 2, and of the fluxion of FE to EQ; consequently (art. 384) if CE be the ray of curvature at E, and Ck be perpendicular to EQ in k, Ek will be to EQ as 2 to $n + 1$, and CI to CE as the difference of 1 and n to 2. But the fluxion of HI is to the fluxion of the curve DE in the ratio compounded of that of CI to CE, and of the ratio of EI to EQ, or of the velocity at D to the velocity at E. Therefore when FH is described with the velocity acquired at D, the fluxion of the time in FI is to the fluxion of the time in the line of swiftest descent as CI to CE, or the difference of 1 and n to 2; and the time in which the right line IH is described by a motion equal to that which is acquired at D, is to the time of descent along the arch ED in the line of swiftest descent in the same ratio. This theorem is not extended to the case wherein n is equal to unit; in which AED is an arch of a circle, and the point I has no motion. What was shown in art. 407, concerning the motion of a body that descends along a cycloid in the common hypothesis of gravity is a particular case of this theorem.

578. In order to discover the nature of the line of swiftest descent, when the gravity is directed towards a given centre, the following lemma will be of the same use as that in art. 572 was in the preceding case. Let AI and KL (*fig.* 257) be circles described from the same centre S; and, the point A being given upon AI, let E be any point upon KL, and SE meet AI in M; join AE, suppose AE to be described with any given velocity represented by u, the arch AM to be described with a given velocity represented by b, and the difference of the times in which AE and AM will be thus described will be least (or $\frac{AE}{u} - \frac{AM}{b}$ will be a *minimum*), when the sine of the angle SAE is to the radius as u is to b, if the ratio of u to b be less than that of SK to SA. Let SH be to SA as u is to b, and SE meet the circle HNk described from the centre S in N, then HN will be to AM as SH to SA or u to b; so that HN will be described with the velocity u in the same time that AM is described

with

with the velocity b. Therefore the difference of the times in which AE is described with the velocity u and AM with the velocity b, is equal to the time in which AE — HN is described with the velocity u, and is least when AE — HN is least. Let AP the tangent of the circle HNh from A meet the arch KEk in L, and Sp be perpendicular to AE in p, then the fluxion of AE will be to the fluxion of KE as Sp to SE or SK; but the fluxion of KE is to the fluxion of HN as SK to SH; consequently the fluxion of AE is to the fluxion of HN as Sp is to SH, and the fluxion of AE — HN is to the fluxion of HN as Sp — SH or Sp — SP to SP. Therefore KE and HN being supposed to increase uniformly, AE — HN decreases till E come to L, where its fluxion vanishes (because Sp becomes then equal SP), and thereafter it increases till AE become a tangent to KEk; consequently AE — HN, or $\frac{AE}{u} - \frac{AM}{b}$, is a *minimum* when E falls upon L, in which case the sine of SAE is to the radius as SP or SH to SA, that is, as u to b. Though this be sufficient for our present purpose, it may be worth while to observe, that if AP produced meet the circle KLk in l, AE — HN is a *maximum* when E comes to l, SH being less than SK; but that when SH is equal to SK, AE — HN (which in this case is AE — KE) never becomes a *minimum* or *maximum*, though its fluxion vanishes when AE becomes a tangent of KEk: and this is an instance of what was shown in art. 261, concerning the inaccuracy of the common rule for determining a *maximum* or *minimum*, and the correction that is requisite to render it general. For the arch HN being supposed to flow uniformly, the fluxion of AE — HN is as Sp — SH, and it is easy to see that the fluxion of Sp — SH or the second fluxion of AE — HN vanishes in this case as well as its first fluxion, but that its third fluxion does not vanish.

579. In the same manner when e is taken upon kl a circle described from the centre S with a radius Sk greater than SA, and Se intersects the arch AI in m, $\frac{Ae}{u} - \frac{Am}{b}$ is a *minimum* when the sine of the angle SAe or SAE is to the radius as u to b;

consequently $\frac{Ee}{u}-\frac{Mm}{b}$ is likewise a *minimum* in this case, *e*E being any right line terminated by the circles KL, *kl*.

580. The gravity being directed towards the centre S, let FED be the line of the swiftest descent from F to any vertical line SDH, AE any arch of FED, and let the rays SA, SE meet the circle DLK described from the centre S in K and L. Then EM the difference of SA and SE being given, the excess of the time of descent in AE the arch of the line of swiftest descent above the time in which the circular arch KL would be described with the velocity acquired at D will be a *minimum*. Let *k*E an arch described from the centre S meet SA in *k*, let A*e* and Ae be any lines drawn from A to this arch, let the times of descent in AE, A*e*, and Ae be represented by T . AE, T . A*e*, and TAe, and drawing S*e*, Se that meet DK in *l* and l, let the times in which KL, K*l*, and Kl would be described by the velocity acquired at D be represented by T . KL, T . K*l*, and T . Kl, I say that T . AE — T. KL will be less than T . A*e* — TK*l* or T . Ae — T . Kl. It is manifest that D must be the lowermost point of FED; for let F*z*D be a line that has its lowermost point at *z*, and *zr* be perpendicular to SD in *r*, then because *zr* would be described in less time than *z*D and F*zr* in less time than F*z*D, F*z*D cannot be the line of swiftest descent from F to the vertical SH. Let *ed* and ed be equal and similar to ED and situated similarly to the rays S*e*, Se as ED is to SE; so that *ld* and ld may be each equal to LD, and *l*L equal to *d*D, and Ll to Dd. Let *e* be betwixt *k* and E, and e on the other side of E, then the time of descent in AED being less than in A*ed*D by the supposition, and the times of descent in *ed* and ED equal, it follows that T . AE is less than T . A*e* + T . D*d* or T . A*e* + T . KL — T . K*l*; therefore T . AE — T . KL is less than T . A*e* — T . K*l*. Let ed meet SH in *x* and the time of descent in AED being less than the time of descent in Ae*x* by the supposition, and the time in which Dd is described by the motion acquired at D less than the time of descent in *x*d, it follows that the time of descent in AEDd is less than in Aed, and by subducting the equal times in ED and ed, it appears that T . AE + T . Dd (or T . Ll, or T . K*l* — T . KL) is less than T . Ae;

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consequently T . AE — T . KL is less than T . Ae — T . Kl. Therefore T . AE — T . KL is a *minimum* when AE is any arch of the line of swiftest descent, and EM or SA — SE is given.

581. The nature of the line of swiftest descent, when the gravity is directed to a given centre, is easily discovered from this property, by art. 578. For since T . AE — T . KL is a *minimum*, it follows that if SE meet the circle AMP described from the centre S in M, and we suppose the arch AM to be described with a velocity *b* which is to the velocity *a* acquired at D as SA is to SD or SK, the time in which AM is thus described will be equal to the time in which KL is described with the velocity *a*; and if the time in which AM is thus described be expressed by T . AM, then T . AE — T . AM will be likewise a *minimum*, EM the difference of SA and SE being given. Therefore (art. 578) it is the nature of the line of swiftest descent in this case, that if AT be the tangent at A, the sine of the angle SAT will be to the radius as the velocity at A is to the velocity *b*, which is itself supposed to be to the velocity acquired at D as SA to SD. That is, the sine of the angle SAT, in which any ray SA intersects the curve at A in the line of swiftest descent is always to the radius in the ratio compounded of the direct ratio of the velocity acquired at A to the velocity at D, and of the inverse ratio of the distance SA to the distance SD. And this is the analysis of the problem when the gravity is directed towards a given centre.

582. Let FED (*fig.* 258) be now a line of such a nature that the sine of the angle contained by the curve FE and ray SE is to the radius in the compound ratio of the velocity at E to the velocity at D and of SD to SE. Let AEB be an arch of this line; let AT, T*t*, and *t*B be the tangents at A, E, and B: let AM, RT, *rt*, and *p*B the circles described from the centre S through A, T, *t*, and B, and SA, ST, S*t*, and SB meet the circle D*a* described from the centre S in *a*, *m*, *n*, and *b*. Let the tangents AT, T*t*, and *t*B be described with the velocities acquired at the respective points of contact A, E, and B; and the excess of the time in which AT*t*B will be thus described above the time in which the circular arch *ab* would be described with the velo-

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city acquired at D, will be less than if the points T and t were taken any where else upon the arches RT and rt. For let the velocities acquired at A and D be called u and a, and b be to a as SA to SD; then since the sine of the angle SAT is to the radius as u is to b, it follows that if AM be described with the velocity b, then T . AT — T . AM will be a *minimum*, TM being given, by art. 578; but AM and am are described by those velocities b and a in equal times; consequently T . AT—T . am is likewise a *minimum*. In the same manner T . TEt—T . mn and T . tB—T . nb are *minima* by art. 579; the differences of the rays ST and St, and of St and SB, being given. Thus, by proceeding as in art. 576, it will appear synthetically that the excess of the time of descent in the arch AEB above the time in which ab is described with the motion acquired at D is a *minimum*, SA — SB being given. Therefore AED is described in less time than any other line that passes through A and D, the velocities being supposed equal at A, and consequently at all other equal distances from S.

583. What (*fig.* 259) we have shown concerning the lines of swiftest descent will be found to agree with the second general principle described above in art. 563, and may be deduced from it. For let PN the ordinate of the figure HNG always represent the velocity at P, or at E, SE being always equal to SP; let SX be perpendicular to the tangent at E in X, and SX will be to PN as SD to DG, by the last article; consequently since the fluxion of SX will be to the fluxion of PN as SX to PN and to the fluxion of SP as SX to PT, if Ed be one half of the chord of the circle of curvature which passes through S, Ed will be equal to PT by art. 384, as it ought to be according to the second general principle in art. 563. And from this property of those lines it may be demonstrated synthetically that AEB is not only the line of swiftest descent from A to B, but from any point in Aa the ray of curvature at A to any point in Bb the ray of curvature at B, providing the curve HNG be concave towards HD. It may be worth while to describe this method, though it is not applicable when HNG is convex towards HD, not only for the further illustration of this subject, but because

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Plate XXIX. *Pa. 84. Vol. II.*

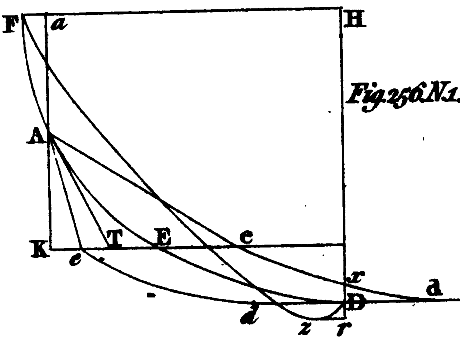
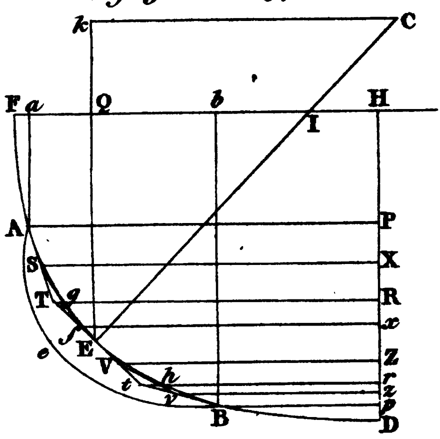


Fig. 256. N. 1.

Fig. 256. N. 2. Art. 576. & 577.



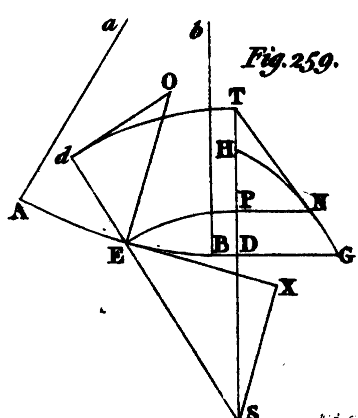


Fig. 259.

it may be applied to other cases than these we have considered hitherto.

584. Let AEB (*fig.* 260) be a line that is described by the evolution of aCb, as in art. 402, let M be any point in the ray CE, and if the velocity that would be acquired at M be to the velocity at E always in a less ratio than CM to CE, then AEB is not only the line of swiftest descent from A to B, but from any point in Aa to any point in Bb. For let the velocity at M be to the velocity at E as any line CZ less than CM is to CE, KML any line bounded by Aa and Bb in K and L, and fMh a line described through M by the evolution of the same curve aCb, so as to be always perpendicular to the ray CE while aCb is evolved. Then the fluxion of the time in AE will be to the fluxion of the time in fM in the ratio compounded of that of the fluxion of AE to the fluxion of fM (or of CE to CM), and of the ratio of the velocity at M to the velocity at E, or of CZ to CE, that is in the ratio of CZ to CM; consequently the fluxion of the time in AE is always less than the fluxion of the time in fM, which is itself never greater than the fluxion of the time in KM; because CE is supposed to be always perpendicular to fM, and the fluxion of fM never can exceed the fluxion of KM. Therefore, the fluxion of the time in AE is always less than the fluxion of the time in KM, so that the time in AE must be less than the time in KM, and the time in AEB less than the time in KML.

585. Supposing the gravity to be directed to the centre S, and to be always of the same force at the same distance from it, let SP be taken upon any given ray SH always equal to SE; and the velocity at any distance SE, or SP, being such as would be acquired by a descent from the distance SH, let PN the ordinate of the figure HNG represent this velocity; and NT the tangent of HNG at N meet SH in T. Let Ed be taken upon SE produced from E always equal to the subtangent PT, and dC perpendicular to Sd meet EC, a perpendicular to the curve AEB in C. Then, if the point C determined in this manner, be always the centre of the curvature of AEB at E, and HNG be concave towards HD, AEB will be the line of swiftest descent from any point in the right line Aa to any point in Bb. For let M be any point upon CE different from E, lMp an arch of

 a circle

a circle from the centre S meet SH in p and SE in l, pn the ordinate at p meet HNG in n and NT in z, and lx perpendicular to Sl meet CE in x. Then the velocity at M will be to the velocity at E as pn to PN, and consequently in a less ratio than pz to PN, or Tp to TP, or dl to dE, or Cx to CE, and therefore (Cx being less than CM) in a less ratio than CM to CE. From which it follows (by the last article) that AEB is described in less time than any other line drawn from any point in Aa to any point in Bb, the velocities being supposed equal at equal distances from S, or being such as would be acquired by a decent from the same distance from the centre S. The same demonstration takes place when HNG is a right line, because in that case Cx is still less than CM. It is not applicable when HNG is convex towards HD, nor is AEB in this case the line of swiftest descent from Aa to Bb; but it appears from art. 582, that it is the line of swiftest descent from the point A to the point B. It is obvious that the same demonstration takes place when the gravity acts in parallel lines and HNG is concave towards HD, by substituting a horizontal line in place of the arch pMl: but it is not applicable when HNG is a right line, in which case an arch of a circle is the line of swiftest descent from A to B, but a similar concentric arch is described in the same time with AEB, as may be easily demonstrated.

586. The figures constructed in art. 392 and 393, which are the *trajectories* and *catenariæ* (by art. 436, 437, 438, and 569), in some of the most simple cases when the centripetal or centrifugal force is inversely as a power of the distance from the centre, of an exponent greater or less than unit, are likewise the lines of swiftest descent in certain analogous cases. When a centripetal force is inversely as a power of the distance n greater than unit, and the velocity at any point F is to the velocity in a circle at the distance SF in the subduplicate ratio of 2 to $n-1$, the line of swiftest descent from F to the vertical SDH is the same that was constructed in art. 392, from the right line AM (*fig.* 171) by taking always the angle ASL to ASM as 2 to $n+1$, and SL to SA as the power of SM of the exponent $\frac{2}{n+1}$ is to the same power of SM. For in these figures the sine of the angle SAT is inversely as the power of SA of the exponent

nent $\frac{1}{2}n+\frac{1}{2}$, and therefore in the compound ratio of the direct ratio of the velocity (which is inversely as the power of SA of the exponent $\frac{1}{2}n-\frac{1}{2}$) and the inverse ratio of the distance; consequently, these are the lines of swiftest descent in this case, by art. 582. For example, when the force is inversely as the cube of the distance (or n is equal to 3), and a body descends from any point F with a velocity equal to the velocity in a circle at the distance SF, the line of swiftest descent AEB is an arch of an equilateral hyperbola that has its centre in S. When n is equal to 2, and the velocity at F to the velocity in a circle at the distance SF as $\sqrt{2}$ to 1, the figure is constructed by taking ASL equal to $\frac{2}{3}$ASM and SL equal to the first of two mean proportionals betwixt SM and SA. When a centrifugal force acts upon the body that is inversely as a power of SA of an exponent n less than an unit, and the velocity at any point F is to the velocity in a circle described at the distance SF by a centripetal force equal to the centrifugal force at F in the subduplicate ratio of 2 to $1-n$, AEB is likewise one of the figures constructed in art. 392, by taking ASL to ASM as 2 to $1-n$, and SL to SA as the power of SM of the exponent $\frac{2}{1-n}$ to the same power of SA. For example, when the centrifugal force is constant, and the velocity at F to the velocity in the circle at the distance SF as $\sqrt{2}$ to 1, the line of swiftest descent is an arch of a parabola that has its *focus* in S. When the centrifugal force is as the distance from S, and the velocity at F equal to the velocity in the circle, the line of swiftest descent from one given point to another is an arch of a circle or of a logarithmic spiral. When the centrifugal force is as a power of the distance of the exponent n greater than unit, and the velocity at F is to the velocity in a circle described at the distance SF with a centripetal force equal to the centrifugal force at F as 2 is to $n+1$, the line of swiftest descent is one of those constructed in art. 393 (*fig.* 172) from the circle, by taking ASL to ASM as 2 to $n-1$ and SL to SA as the power of SM of the exponent $\frac{2}{n-1}$ to the same power of SA. And, in this case according as n is equal to 2, 3,

or 5, AEB is an arch of an epicycloid described by a point in the circumference of a circle that revolves on an equal circle, or it is an arch of a semicircle, or of the *lemniscata*. It is obvious that these figures satisfy likewise the problem, when a body which is acted upon by a force diffused from S that is as any power of the distance, moves from a given point A to a given point E with a given velocity, and it is required that the sum of the actions shall be a *minimum*. For example, if the power be inversely as the square of the distance, AE is an arch of a semicircle described through the three points S, A, and E; if it be inversely as the power of the distance of the exponent m greater than unit, then AMS being a semicircle let the angle ASL (*fig.* 172) be to ASM as 1 to $m-1$, and SL to SA as the power of SM of the exponent $\frac{1}{m-1}$ to the same power of SA.

587. To conclude this subject with an instance which may show the extensive use of the second general theorem in art. 568, and how it may serve for finding the curvature of the line of swiftest descent when the gravitation tends to several centres, let equal and uniform forces tend towards C and S as in art. 491. Let the velocity (*fig.* 217) be such as would be acquired by a descent from F, E any point in the line of swiftest descent, upon CE produced take Eu equal to the excess of CF+SF above CE+SE, and uz perpendicular to Cu shall intersect the right line Ez, that bisects the angles CES, in z, so that Ez shall be one fourth part of that chord of the circle of the same curvature with the line of swiftest descent at E which bisects the angle CES.

588. Suppose now that it is required to find the nature of the line that of all those that pass through two given points A and B (*fig.* 256), and have equal perimeters, is described in the least time. Let the time in which AEB is described by a body that descends along it by its gravity be expressed by T . AEB, and the time in which the same line would be described uniformly with any given velocity a by t . AEB, and let the ratio of m to n be any given ratio; then if $\frac{m}{n} \times$ T . AEB $\mp$ t . AEB be a *minimum*, AEB will be described in less time than any other line

line AeB of an equal perimeter. For let $T.AeB$ represent the time of the descent along AeB, t, AeB the time in which AeB would be described uniformly with the same given velocity a; then by the supposition $\frac{m}{n} \times T.AEB \mp t.AEB$ is less than $\frac{m}{n} \times T.AeB \mp t.AeB$. But $t.AEB$ is equal to $t.AeB$, these being the times in which equal lines are described by equal uniform motions. Therefore $T.AEB$ is less than $T.AeB$, that is, AEB is described in less time than any other line of an equal perimeter that passes through A and B. It is obvious that $\frac{m}{n} \times T.AEB + t.AEB$ cannot be a *minimum*, when AEB is greater than the line which is described in less time than any other line whatever that passes through A and B; and that $\frac{m}{n} \times T.AEB - t.AEB$ cannot be a *minimum*, when AEB is less than that line.

589. Let the velocity at any point E be represented by u, and b be to a as m is to n; let V be to u as a is to $b \mp u$, and if AEB be the line that is described in less time than any other whatsoever that passes through A and B, by a velocity which at any point E is represented by V, the same line AEB will be described in less time than any other of an equal perimeter that passes through A and B, by a velocity which at any point E is represented by u: for the fluxion of $T.AEB$ is expressed by the ratio of the fluxion of AE to u, the fluxion of $t.AEB$ by the ratio of the fluxion of AE to a; consequently the fluxion of $\frac{m}{n} \times T.AEB + t.AEB$ by the ratio of the fluxion of AE to V. And $\frac{m}{n} \times T.AEB \mp t.AEB$ is equal to the time in which AEB is described with a velocity that is always equal to V at any point E. Therefore when this time is a *minimum*, $\frac{m}{n} \times T.AEB \mp t.AEB$ is likewise a *minimum*, and AEB is described in less time than any other line of the same perimeter that passes through A and B.

590. Suppose

590. Suppose the gravity to act in parallel lines, which is the case considered by Mr. *Bernouilli;* and if the sine of the angle FEQ be to the radius (or the fluxion of the base FQ be to the fluxion of the curve FE) as V is to an invariable velocity, that is in a ratio compounded of the ratio of u the velocity at E to the sum or difference of b and u and of an invariable ratio, then AEB will be described in less time than any equal line that passes through A and B, by art. 576, which coincides with the equation of the curve that was found by that learned author, by resolving the element of the curve into three infinitely small rectilineal parts and computations from second fluxions. *Mem. de l'Acad. Royale des Sciences,* 1718, prop. 4, and schol. 2.

591. The same method discovers the property of those lines, when the gravity is directed towards a given centre S with the same facility. Let the velocity of the body that descends along the curve AEB at any point E be represented by u, and b express an invariable velocity; then if the sine of the angle SEB (*fig.* 257), contained at E by the curve AE and ray SE, be always to the radius in a ratio compounded of that of u to the sum or difference of b and u, and of the ratio of an invariable line to the distance SE, the line AEB will be described in less time than any equal line that can be drawn from A to B, the velocities being supposed equal at A in each. In general, let EC be half of the chord of the circle of curvature at E, that is in the direction of the force EK that acts upon the body at that point, as in art. 563, and suppose the body to descend from E in CE with the velocity u acquired in the curve at E, then the point C being supposed to remain, if the curvature of the line is such, that the fluxion of u be to the fluxion of EC in the compound ratio of u to EC and of $b \mp u$ to b, then AEB will be described in less time than any equal line that passes through A and B, the velocities being equal at A. The demonstration is similar to that in art. 565.

592. The celebrated isoperimetrical problems may be treated in the same manner, and rendered more general than is usual, without having recourse to the fluxions of the higher orders; and the solutions of these problems (that are generally considered

dered as of a very abstruse nature) may be verified by easy synthetic demonstrations. The lemma that is required for this purpose differs not materially from that in art. 572, which we demonstrated without having recourse to fluxions. Let KL (*fig.* 261) be a right line given in position, A any given point that is not in KL, AK perpendicular to KL in K, E any point upon KL; and let a and u represent any given or invariable lines: then if KL be to AL, or the sine of the angle KAL to the radius, as u is to a, $AE \times a - KE \times u$ will be a *minimum* when E falls upon L. For let KH and EV be perpendicular to AL in H and V, and AR made equal to AE; because KE is to HV as AL to KL, or a to u, $KE \times u$ is equal to $HV \times a$, $AE \times a - KE \times u$ equal to $\overline{AE - HV} \times a$, or $\overline{AH + VR} \times a$, which is evidently least when VR vanishes, that is when E falls upon L. It follows from this that the point A, the distance AK, and Ek the distance of the parallels KL and kl, being given, Ek being perpendicular to kl in k, and E and e being any points upon these parallels; if a, u, and v be supposed invariable, then $AEe \times a - KE \times u - ke \times v$ will be least when the sine of the angle KAE is to the radius as u to a, and at the same time the sine of kEe to the radius as v to a; for in this case $AE \times a - KE \times u$ will be a *minimum*, and $Ee \times a - ke \times v$ will be less than when the angle kEe is of any other magnitude, so that their sum will be less than if the right lines AE and Ee were inflected in any other manner.

593. It is easily demonstrated from what was shown in the last article, that kC (*fig.* 262) and CG being perpendicular to each other in C, KMNLG a given figure applied on CG, E any point in the curve AED, EPN a parallel to kC that meets CG in P, and the curve ML in N, Epn a parallel to CG that meets kC in p, upon which pn is supposed to be taken always equal to PN the ordinate of the figure KMNLG, and to generate the area *kmnlg* in this manner, then if the point A, the distance KG (or the difference of Ak and Dg the ordinates from the curve AED to fC), and the figure KMNLG with the right line a be given, the excess of $AED \times a$ above *kmnlg*, the area generated by pn, will be a *minimum*, when the sine of the angle AEp contained by the curve AE and any ordinate Ep, is to the radius as PN the

the corresponding ordinate of the figure KMNLG is to the invariable line *a*. For let TE*t* the tangent at E meet AT the tangent at A in T, and D*t* the tangent at D in *t*, let TRS and *t*VX parallel to *k*C meet CG in R and V, and ML in S and X, T*rs* and *tvx* parallel to CG meet *k*C in *r* and *v*, and *mnl* in *s* and *x*; complete the parallelograms *krsu*, *rpno*, *pnyv*, *vxzg*; and the excess of ATE*t*D × *a* above the figure *kusoyxzg*, the sum of these parallelograms, will be less than if the points T, E, *t*, D were taken any where else upon the parallels TR, EP, *t*V, DG, by the last article. Then by drawing tangents to the curve AEB at the points where TR and *t*V intersect it, and parallels to *k*C through the points where these tangents intersect AT, T*t*, and *t*D, and proceeding in this manner, the ultimate ratio of the circumscribed figure AT*t*D to the curve AED, and of the area *kusoyxzg* to the curvilineal area *kmnlg*, will be a ratio of equality; consequently AED × *a* — *kmnlg* will be a *minimum*; the point A, with the right lines *a* and KG, and the figure KMLG being given.

594. It follows that the point A being given with the figure KMLG, if *pn* be always equal to PN, and the sine of the angle AE*p* be always to the radius as PN to *a*, then the area *kmnlg* will be greater than any area *kmvlg* generated in the same manner from any line AED that is drawn from A to LGD of a perimeter equal to AED. For since AED × *a* — *kmnlg* is less than A*e*D × *a* — *kmvlg*, by the last article; and A*e*D is equal to AED, by the supposition; it is manifest that *kmnlg* must be greater than *kmvlg*. Therefore when the sine of the angle AE*p* is always to the radius as PN to *a*, AED is the line that amongst all those which are drawn from A to LGD, and have equal perimeters, produces the greatest area *kmnlg*.

595. The rest remaining as before, let HI parallel to KG at any given distance KH meet EP in Q, and *hi* parallel to *kg* at an equal distance *kh* meet E*p* in *q*, and the points *h*, *k*, *m* lie in the same order from each other as H, K, M. Then *qn* will be always equal to QN, and the area *hinm* generated by the ordinate *qn* will be a *maximum* or *minimum* according as HI and KG are on the same or different sides of MNL, the points A and D being given. For since *kmnlg* is always a *maximum*, and the

the rectangle *hg* is given, *hmnli* will be a *maximum* when *hi* and *kg* are on the same side of *mnl*, that is when HI and KG are on the same side of MNL; and *hmnli* will be a *minimum* when *hi* and *kg* are on different sides of *mnl*, that is when HI and KG are on different sides of MNL. It appears therefore that the area *hmnli* generated by the ordinate *qn*, equal to QN, is a *maximum* when the sine of the angle AE*q* is always to the radius (or the fluxion of the base *hq* to the fluxion of the curve AE) as QN + HK is to *a*, or as QN — HK to *a*, if QN in this latter case be never less than HK; because when HI and KG are on the same side of MNL, PN (or *pn*) is either equal to QN+HK, or to QN—HK, QN being in this case never less than QP or HK; but that the area *hmnli* is a *minimum*, when the sine of the angle AE*q* is always to the radius (or the fluxion of the base *hq* to the fluxion of the curve AE) as HK—QN to *a*, HK being supposed to be never less than QN any ordinate from MNL to the axis HI; because when KG and HI are on different sides of MNL, HK—QN is equal to PN. The points A and D with the figure HMLI are supposed to be given, and the perimeter AED to be always the same. And this property of the line AED, which amongst all those of an equal perimeter that pass through A and D produces the greatest or least area *hmnli*, by taking the ordinate *qn* always equal to QN the corresponding ordinate of the figure KMNLG, is the same that Mess. *Bernouilli*, Dr. *Taylor*, and others, deduced from computations that involve third fluxions. It is obvious that the curve AED is concave, or convex, towards AK, according as the sine of the angle AE*q* increases, or decreases, while KP increases; that is, according as PN increases, or decreases, while KP increases. When HI intersects MNL in any point betwixt M and L, *hi* intersects *mnl* at some point betwixt *m* and *l*; and one part of the figure AED produces a *maximum* and the other a *minimum*. When MNL meets KG, the angle AE*q* vanishes, and the curve touches the ordinate E*q*; and if PN become equal to *a*, AE*q* will become a right angle, and the curve perpendicular to the ordinate.

596. Because the sine of the angle contained by the curve and ordinate E*q* is to the radius as PN to *a*, it follows from

art.

art. 576, that AED is likewise the line of swiftest descent when the velocity in any part of the right line EPN is always measured by the corresponding ordinate PN, as was observed by Mr. *James Bernouilli*; and this analogy between the lines that satisfy these two problems is accounted for, from the similitude of the methods by which their properties were demonstrated in art. 576 and 595.

597. Suppose now that S (*fig.* 263) is a given point, that a circle *gpk* is described from the centre S with a given radius S*g*, that SA, SE, and SD meet this circle in *k*, *p*, and *g*, that the figure HMLD being given, and SQ being always equal to SE, S*n* is taken upon SE always equal to QN the corresponding ordinate of HMLD; and that S*m* and S*l* being taken upon SA and SD respectively equal to HM and DL the corresponding ordinates of the same figure, it is required to find the nature of the line AED that amongst all those which pass through the points A and D, and have equal perimeters, produces in this manner the greatest or least area S*mnl*. It is manifest that *a* being an invariable line, if AED × *a*—*kmlg* be a *minimum*, then *kmlg* will be greater than any area formed in the same manner from any figure that has its perimeter equal to AED; and that S*mnl* will be the greatest or least of the areas terminated by S*m* and S*l* according as the point S and the circular arch *kpg* are on the same or on different sides of *mnl*. To find when AED × *a*—*kmlg* is a *minimum*, let AT the tangent at A meet the circle QE described from the centre S in T, and ST produced meet the arch ARH described from the same centre in R, and it appears from art. 578, that AT × *a*—AR × *u* is a *minimum* (HQ being given) when the sine of the angle SAT (in which any ray SA intersects the curve AED) is to the radius as *u* to *a*. Let us therefore suppose AR × *u* to be ultimately equal to the area *kmnp*; and since *kmnp* is ultimately to the sector SAR (or ½SA × AR) as the difference or sum (according as *k* and *m* are on the same or different sides of S) of the squares of S*m* and S*k* to the square of SA, it will follow that *u* is to ½SA as the difference or sum of those squares is to SA^2; so that *u* is to *a*, and consequently the sine of the angle SAE is always to the radius, as that difference or sum is to 2SA × *a*. Therefore in the figure AED, if SX be perpen-

perpendicular to the tangent EX at X, Sg and QN be represented by c and V, respectively, the rectangle $2a \times SX$ will be equal to the difference or sum of VV and cc. The invariable quantities a and c (with another invariable line that will arise in determining the figure from this property) serve to satisfy the conditions of the problem, which requires that the curve shall pass through A and D, the perimeter being supposed of the same magnitude in all these figures amongst which AED produces the greatest or least area *Slnm*.

598. When Sg is supposed to vanish, $2a \times SX$ is equal to QN^2, or SX is a third proportional to $2a$ and QN, and the area *Smnl* is a *maximum*. In this case, if HMNLD be a parabola that has its vertex in S and axis in SH, or an hyperbola of any order whose ordinate QN is inversely as any power of SQ, AED is one of the figures constructed in art. 392 or 393, which we have found already to satisfy the most simple cases of several problems in art. 436, 437, 438, 439, 567, and 586. For example, when Sn is taken upon SE always equal to a mean proportional betwixt SE and a given line, and AED is an arch of a logarithmic spiral that has its pole in S, the area *Smnl* is the greatest that can be produced in the same manner from an arch of an equal perimeter that passes through A and D. When Sn is inversely in the subduplicate ratio of SE, and AED is an equilateral hyperbola that has its centre in S, *Smnl* is the greatest area that can be produced in the same manner, from any arch of an equal perimeter that passes through A and D. When MNL is a right line that passes through S, AED is an arch of a circle, and *mnl* is likewise an arch of a circle similarly situated with respect to S, whether Sg be supposed to vanish or not; and, in this case, it is well known that the area *Smnl* is a *maximum* or *minimum* according as *mnl* is concave or convex towards S.

599. It appears, in the same manner, that if any given line *hqi* meet SA, SE, and SD in *h*, *q*, and *i*; and *qn* be always taken upon Sq equal to QN, it will be the property of the figure AED, that amongt those of an equal perimeter which pass through A and D, produces the greatest or least area *hmli*, that

the

the sine of the angle AEX will be to the radius as the difference or sum of the squares of $\overline{Sq+QN}$ and Sg to $2SE \times a$.

600. The property of the line AED that is described by a velocity which at the distance SE, or SQ, is always measured by QN, the ordinate of a given figure HMLD, and that of all those which pass through A and D, and are described in the same time, comprehends the greatest or least area SAED, is that the sine of the angle SEX is to the radius in the compound ratio of QN to an invariable velocity, and of the difference or sum of the square of SE and an invariable square to the rectangle contained by SE and an invariable line. For the construction being the same as in art. 597, if $aa \times T$. AED—AEDgpk be a *minimum*, the point A and right line HD being given, the area AEDgpk will be greater than any area AeDgpk which is terminated by any line AeD that is described in the same time with AED; and SAED will be greater or less than SAeD, according as the point S and arch kpg are on the same or different sides of AED. Because $\frac{AT}{u} - \frac{AR}{b}$ is a *minimum* (HQ being given) when the sine of SAT is to the radius as u is to b, by art. 578, let QN be equal to u, and suppose $\frac{AEpk}{aa}$ ultimately equal to $\frac{AR}{b}$. Then because AEpk is ultimately to the sector SAR, or $\frac{1}{2}SA \times AR$, as the difference or sum of SA^2 and Sk^2 to SA^2; it follows, that b will be to 2SA, as aa is to that difference or sum; and that u is to b, or the sine of the angle in which any ray SA intersects the curve to the radius, in the compound ratio of the same difference or sum to aa and of QN to 2SA. When the gravity acts in parallel lines, the nature of the line AED is discovered in the same manner.

601. The following *lemma* leads to an easy solution of the second general isoperimetrical problem. The point A (*fig.* 264) being given, the right lines AE, b, and u being given in magnitude, and AG given in position, let EK be perpendicular to AG in K, and $EK \times b + AK \times u$ will be a *maximum*, when the tangent of the angle AEK is to the radius as u is to b. For let the circle BEb described from the centre A with the given radius AE meet AG in G; let Gg be erected perpendicular to AG, so that

that Gg and KE may be on different sides of AG, and let Gg be to AG as *u* is to *b*; join A*g*, and A*g* will be given in position; let EK produced meet A*g* in O, then because KO is to AK as Gg to AG or *u* to *b*, AK × *u* will be equal to KO × *b*; so that EK × *b* + AK × *u* will be equal to EO × *b*, and will be a *maximum* when EO is greatest; and it is manifest that EO is greatest, when ET the tangent of the circle at E is parallel to A*g*, or when AE is perpendicular to A*g*, that is, when AK is to KE, or the tangent of the angle AEK to the radius, as G*g* to AG, or as *u* to *b*. In the same manner it appears, that if the right lines E*e* and *v* be likewise given in length, and *ek* be perpendicular to AG in *k*, then *ek* × *b* + AK × *u* + K*k* × *v* will be a *maximum* when the tangent of the angle AEK is to the radius as *u* to *b*, and at the same time the tangent of E*ek* to the radius as *v* to *b*; because EK × *b* + AK × *u* is then a *maximum*; and, E*m* parallel to AG being supposed to meet *ek* in *m*, *em* × *b* + E*m* × *v* will be greater than if the angle E*ek* was of any other magnitude. In general, let the line AE*ef*D consist of any number of parts AE, E*e*, *ef*, *f*D whereof each is given in length; let EK, *ek*, *fl*, DL be perpendicular to AL; and upon these perpendiculars take K*u*, *kv*, *lx*, L*z* equal to any given lines; then the radius being supposed equal to *b*, let the figure of the line AE*ef*D be such that the tangents of the angles AEK, E*ek*, *efl*, *f*DL may be respectively equal to the perpendiculars K*u*, *kv*, *lx*, L*z*; complete the parallelograms AK*ur*, K*kvs*, *klxt*, *l*L*zy*; and the sum of the rectangle LD × *b* added to the area A*rufvtxyz*L (which is made up of those parallelograms) will be greater than if the line AE*ef*D was disposed into any other figure.

602. Let AEDL (*fig.* 265) and ABMIL be applied upon the same axis AL, EPM parallel to DL meet BMI in M, and the tangent of the angle AEP, in which the curve AED intersects its ordinate EP, be always to the radius as PM to *b*. Let A*ed* be any other line equal in length to AED, and the arch A*e* being always equal to AE, let *epm* parallel to DL meet AL in *p*, and *pm* always equal to PM generate the area AB*mil*; then it follows from what was shown in the last article, that DL × *b* + ABMIL will be always greater than *dl* × *b* + AB*mil*. From

which it follows, that when dl is equal to DL, the area ABMIL is greater than ABmil. And if we suppose d to coincide with D (and consequently il coincide with IL), and, AH being given upon AB, if HG parallel to AL meet ID in G, PM in Q, and pm in q, the area HBMIG generated by the ordinate QM will be greater or less than HBmIG generated by the ordinate qm, according as HG and AL are on the same or different sides of MNI. Therefore since QM is equal to the sum or difference of PM and AH, it is the property of the line AED, which of all those that pass through the points A and D and have equal perimeters, produces the greatest or least area HBMIG, when the ordinate QM depends upon the length of the arch AE, being what is called a *function* of the arch AE (that is, QM being always equal to the ordinate of a given figure when the base is equal to AE), that the tangent of the angle AEP is to the radius, or the fluxion of the base AP to the fluxion of the ordinate PE, as the sum or difference of QM and an invariable line AH is to an invariable line b. And this agrees with what the authors above-mentioned found by their computations when carried on justly.

603. It appears as in art. 597 (*fig.* 266), that when S is a given point, and upon SE a right line SM is always taken equal to a *function* of the arch AE (that is, equal to the ordinate of any given figure when the base is equal to the arch AE), and the area SBML is the greatest or least of all those that can be thus produced by lines of equal perimeters that pass through A and D, the tangent of the angle SED, in which any ray intersects the curve, is always to the radius as the difference or sum of the square of SM and an invariable square to $2SE \times b$.

604. The other isoperimetrical problems may be reduced to these, or treated in like manner. For example, let ER parallel to AK (*fig.* 267) meet AS parallel to KD in R, and RN, SV the ordinates of the figure ANVS be always equal to *functions* of the arches AE and AED; let NZ and VX be perpendicular to KA produced in Z and X. Then because the area AVX is equal to $AS \times SV$ — ANVS, and when A and D are given, and the arch AED is given in length, its *function* SV being given, the rectangle $AS \times SV$ is given, it follows that the area AVX is a *maxi-*

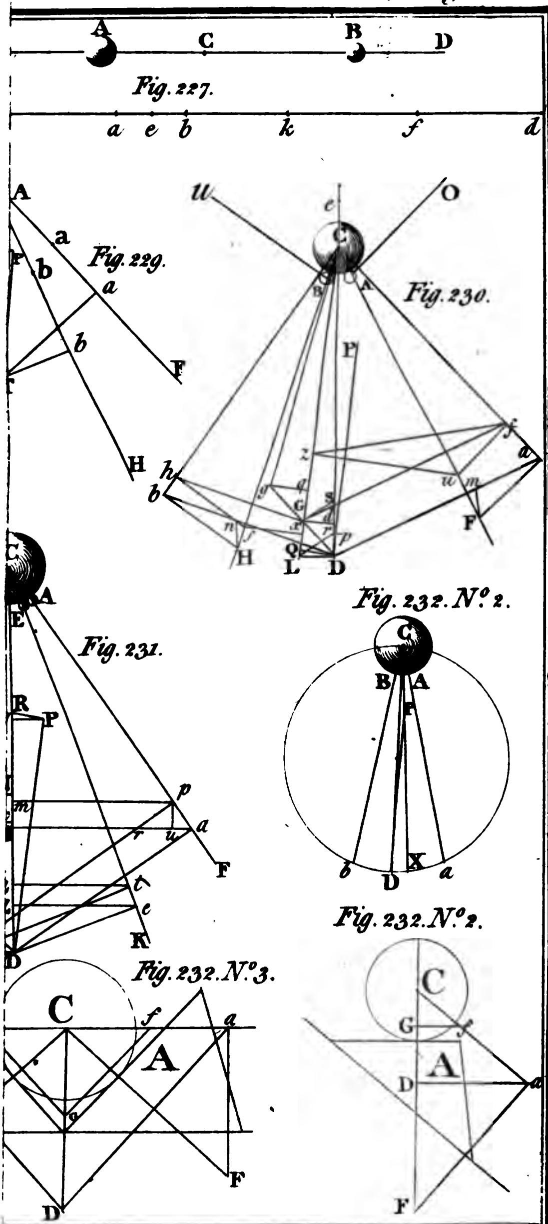


Fig. 227.

Fig. 229.

Fig. 230.

Fig. 231.

Fig. 232. N°. 2.

Fig. 232. N°. 2.

Fig. 232. N°. 3.

a *maximum* when AVS is a *minimum,* and that AVX is a *minimum* when AVS is a *maximum,* that is (by art. 602), when the tangent of the angle AER is to the radius as the sum or difference of RN and an invariable line is to an invariable line. The area AVX has its ordinate ZN equal to EP the ordinate of AED, and its base AZ equal to RN a *function* of AE; and the line AED is the figure which is assumed by a chain perfectly flexible, and suspended from A and D, when the thickness of the chain at any point E is as the fluxion of RN; because the centre of gravity of such a chain would descend to as low a place as possible.

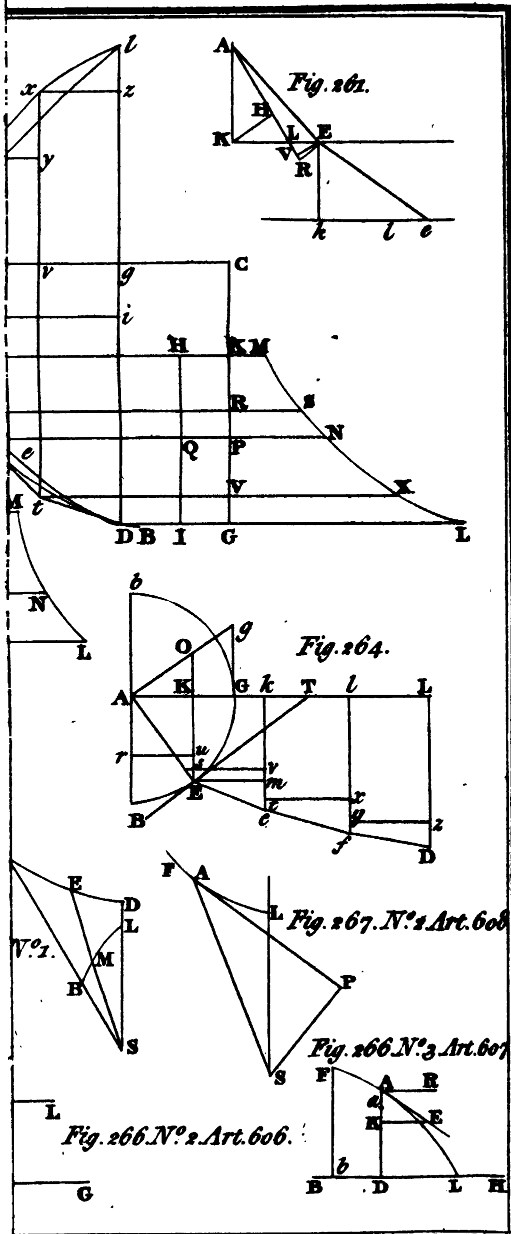
605. The rest remaining as in art. 572 (*fig.* 254), let us now suppose that u the velocity with which AE is described is not given, but varies as a power of AE whose exponent is any number n. And when the sine of the angle KAE is to the radius as u is to $\overline{1-n} \times a$, $\frac{AE}{u} - \frac{KE}{a}$, or $\frac{AE}{u} + \frac{KE}{a}$, is a *minimum* according as n is less or greater than unit. For $\frac{AE}{u}$ will be as AE^{1-n}, and the fluxion of $\frac{AE}{u}$ to the fluxion of AE (art. 167) as $\overline{1-n} \times \frac{AE}{u}$ to AE, or as $1-n$ to u. The fluxion of AE is to the fluxion of KE as KE to AE by art. 193, consequently the fluxion of $\frac{AE}{u}$ is to the fluxion of $\frac{KE}{a}$ as $\overline{1-n} \times KE$ to $AE \times \frac{u}{a}$. Therefore if n be less than unit, these fluxions are equal, and $\frac{AE}{u} - \frac{KE}{a}$ is a *minimum,* or its fluxion vanishes, when KE is to AE as u is to $\overline{1-n} \times a$. And when n is greater than unit, $\frac{AE}{u}$ decreases while $\frac{KE}{a}$ increases, and $\frac{AE}{u} + \frac{KE}{a}$ is a *minimum,* or its fluxion vanishes, when KE is to AE, or the sine of KAE is to the radius, as u is to $\overline{n-1} \times a$. In these cases therefore likewise the sine of the angle KAE is still as u; and this theorem thus extended will serve for resolving problems concerning

the *maxima* and *minima*, to which the lemma in art. 572 does not reach.

606 (*Fig.* 266). It remains now to show how the problem concerning the solid of least resistance may be resolved by first fluxions. The point A being given, let the ordinate AD meet KL (parallel to the axis DG) in K, let E be any point upon KL, join AE; and the resistance of the fluid being represented by a given right line AR, the resistance of the right line AK moving in the direction KL will be represented by $AK \times AR$. Let RM be perpendicular to AE in M, and MN perpendicular to AR in N; and the resistance which the conical surface generated by AE (when the figure is supposed to revolve about the axis DG) meets with, will be to the resistance of the annular space generated by AK as RN to AR, and therefore (AK being bisected in *a*) that resistance will be as $Da \times AK \times RN$. Because, AK being given, RN decreases while KE increases, let us enquire when $Da \times AK \times RN + KE \times AR \times a$ is a *minimum*. The fluxion of this sum vanishes, AR (which measures the direct resistance of the fluid) with AK, Da and a being supposed invariable, when the fluxion of KE is to the fluxion of RN as $AK \times Da$ to $AR \times a$. But RN being to AR as AK^2 to AE^2, RN is inversely as AE^2, and (art. 167) the fluxion of RN is to the fluxion of AE as 2RN to AE; consequently the fluxion of KE is to the fluxion of AE, or (art. 193) AE to KE as $2Da \times AK \times RN$ to $a \times AE \times AR$; that is, as $2Da \times AK^3$ to $a \times AE^3$. Therefore $Da \times AK \times R + KE \times AR \times a$ is a *minimum* when $2Da \times AK^3 \times KE$ is equal to $a \times AE^4$.

607. From this it follows, that KE parallel to the axis BH being supposed to meet the ordinate AD in K, and AE the tangent at A in E, and a being an invariable quantity, if the line FAL be of such a nature that $AD \times AK^3 \times KE$ be always equal to $\frac{1}{2}a \times AE^4$, then the solid generated by FAL*b* revolving about the axis BH will meet with less resistance, when it moves in a given fluid with a given velocity in the direction of the axis BH, than the solid generated in the same manner by any other figure whose perimeter passes through F and L. For when AK is continually diminished, Da is ultimately equal to DA;

Plate XXX. Pa 100. Vol. II.



DA; and it follows from the last article that the sum of the solid which measures the resistance of the conical surface generated by AE about the axis BH, added to KE × AR × *a* will be always ultimately a *minimum*. And because the sum of the resistances of these conical surfaces is ultimately equal to the resistance of the solid generated by FAL about the same axis BH; and the sum of the solids KE × AR × *a* is ultimately a given solid *b*L × AR × *a* (because *b*L, AR, and *a* are supposed to be given), it appears that the resistance of the solid generated by FAL is a *minimum*. It is easy to see that this agrees with the property of this solid, which was given by Sir *Isaac Newton*.

608. In the same manner when a plane figure FAL (*fig*. 267) revolves about a given centre S, that is in the plane of the figure, in a medium that resists in the duplicate ratio of the velocity, it is the property of the line which in this case meets with the least resistance, that the sine of the angle SAP contained by the ray SA and tangent at A is inversely as the cube of the tangent AP, SP being perpendicular to AP in P. There are several other enquiries of this nature which might be prosecuted in the same manner, but we proceed to what may be of more use in philosophy.

CHAP XIV.

Of the Ellipse considered as the Section of a Cylinder. Of the Gravitation towards Bodies, which results from the Gravitation towards their Particles. Of the Figure of the Earth, and the Variation of Gravity towards it. Of the ebbing and flowing of the Sea, and other Enquiries of this Nature.

609. THE properties of the circle demonstrated by *Euclid*, *Pappus*, *Greory a St. Vincentio*, and others, suggest analogous properties of the ellipse; which, generally speaking, are most easily and briefly deduced by considering it as the oblique section of a cylinder, or as the projection of a circle by parallel

rays upon a plane that is oblique to the plane of the circle. For the centre of the circle by this projection gives the centre of the ellipse; any diameters of the circle that are perpendicular to each other with the tangents at their extremities (which form the circumscribed square) and their respective ordinates, give conjugate diameters of the ellipse with the circumscribed parallelogram, and the ordinates of these diameters; any parallel lines in the plane of the circle are projected by parallel lines in the plane of the ellipse, that are to each other in the same ratio as those of which they are the projections; any area in the plane of the circle gives by its projection an area in the plane of the ellipse, which is always to the area in the plane of the circle as the transverse axis of the ellipse to its second axis, the cylinder being supposed upright; and any concentric circles give similar concentric ellipses. Having found this method very convenient on these accounts for discovering the properties of the ellipse, and particularly some that are of use in the following enquiries, it may be worth while to prosecute it a little farther than the illustrious *Marquis de l'Hospital* has done, *lib.* 6, *sect. conic.* the rather that it is not difficult to derive the analogous properties of the hyperbola and parabola from those of the ellipse when known.

610. Let $aABb$ (*fig.* 268) be a section of an upright cylinder through its axis cC, $adbe$ a section of this cylinder perpendicular to cC, ADBE a section perpendicular to the plane $aABb$, but oblique to the axis cC; the former will be a circle that has its centre in c; and the latter will be an ellipse that has its centre in C, of which AB will be the transverse axis, and the second axis DE perpendicular to AB will be equal to ab the diameter of the circle. For let h be any point in the circumference of the circle, hp perpendicular to ab in p, hH and pP parallel to cC meet the plane ADBE in H and P, so that H may be what we call the *projection* of h, and P of p; join HP, and it will be perpendicular to the plane $aABb$, consequently $hHPp$ is a parallelogram, and HP equal to hp. Therefore the square of HP is equal to the rectangle apb which is to APB (because ap is to AP, and pb to PB, as ab to AB) as ab^2 to AB^2; consequently AHB is an ellipse, and its second axis DE is equal to ab. It

appears

appears in the same manner, that any right line in the plane of the circle that is perpendicular to *ab*, is projected by an equal right line in the plane of the ellipse perpendicular to AB.

611. Any parallels *gh* and *kl* (*fig.* 269) in the plane of the circle are projected by parallel lines GH and KL in the plane of the ellipse that are in the same ratio to each other as *gh* and *kl*. For the planes *g*GH*h*, *k*KL*l* being parallel, the sections of those planes with ADBE are parallel; and because the angles of the figures *g*GH*h*, *k*KL*l* are respectively equal to each other, GH will be to *gh* as KL to *kl*.

612. It is obvious, that according as *f* is a point without or within the circle, its projection is without or within the ellipse. Therefore the tangent of the circle at any point *h* is projected by the tangent of the ellipse at H. Hence any right line VR parallel to the tangent at H is bisected in M by the diameter that passes through H, and VM, MR are the ordinates of that diameter, being projected from *vm* and *mr* equal perpendiculars to *ch*. Let *vt* the tangent of the circle at *v* meet *ch* produced in *t*, and VT the tangent of the ellipse at V meet CH in T. Then because *m*M, *h*H, and *t*T are parallel, and *cm* is to *cv* as *cv* (or *ch*) to *ct*; it follows that CM, CH, and CT are in continued proportion. Let *tv* produced meet the semidiameter *cl* perpendicular to *ch* in *z*, and because the rectangle *tvz* is equal to the square of *cv* or *ck*, it follows that if TV meet CL the semidiameter conjugate to CH in Z, the rectangle TVZ will be equal to the square of CK the semidiameter of the ellipse that is parallel to TZ. And if HNQ parallel to CK meet CV and CL in N and Q, the rectangle NHQ will be equal likewise to CK^2.

613. Any right line *gh* in the plane of the circle is to GH its projection in the plane of the ellipse as the second axis of the ellipse is to the diameter that is parallel to GH; because this diameter is the projection of the diameter of the circle which is parallel to *gh*; and parallel lines in the plane of the circle are in the same ratio as their projections in the plane of the ellipse, by the last article.

614. Hence right lines *pm* and *pn* (*fig.* 270) in the plane of the circle that form equal angles with *ab*, or with *phg* any parallel to *ab*, on the same or on different sides of that parallel,

are projected by right lines PM and PN that form equal angles with AB the axis of the ellipse, or with PHG the projection of *phg;* and PM is to PN as *pm* to *pn*. For let *mr* perpendicular to *pg* in *r* meet *pn* in *f*, and *mr* will be equal to *fr;* consequently their projections MR and FR will be equal by art. 610, and the angle MPR equal to NPR; and because the diameter parallel to PM is equal to the diameter parallel to PN, it follows that PM is to PN as *pm* to *pn*, by the last article. If *nq* be perpendicular to *pg*, and NQ to PG, PQ will be to PR as *pq* to *pr*, and PQ+PR to *pq*+*pr* as PQ to *pq* or as AB to DE. The use of this property will appear afterwards, but we will first show how other properties of the ellipse are briefly deduced in like manner.

615. If any line VR (*fig.* 271) terminated by the ellipse in V and R meet any parallels GH and KL in M and N, and VR, GH, and KL be projected from *vr*, *gh*, and *kl* in the plane of the circle, GM will be to KN as *gm* to *kn* (art. 611), and MH to NL as *mh* to *nl;* consequently the rectangle GMH will be to KNL as *gmh* to *knl*. In the same manner the rectangle VMR will be to VNR as *vmr* to *vnr*. But *gmh* is equal to *vmr* (*elem.* 35. 3), and *knl* to *vnr*. Therefore the rectangle GMH is to KNL as VMR to VNR.

616. When *hc* and *cl* (*fig* 272) are perpendicular semidiameters of the circle, let *hp* and *lq* be perpendicular to *ab* in *p* and *q*, and let HP and QL be the projections of these ordinates in the plane of the ellipse. Then HP will be equal to *hp* (art. 610), or *cq*, and LQ equal to *lq* or *cp*. Because CP^2 is to cp^2, and CQ^2 to cq^2, as CA^2 to ca^2, it follows, that $CP^2+CQ^2+cp^2+cq^2$, or CH^2+CL^2, is to cp^2+cq^2, or ca^2, as CA^2+ca^2 to ca^2; consequently CH^2+CL^2 is equal to CA^2+ca^2, or CA^2+CD^2; that is, the sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axis AB and DE.

617. Any rectangle *ghlk* (*fig.* 273) in the plane of the circle that is contained by right lines one of which *gk* is parallel and the other *gh* perpendicular to *ab*, is to its projection GHLK in the plane of the ellipse (which is likewise a rectangle) as the second axis DE to the transverse AB. For *gh* and GH the sides of those rectangles perpendicular to *ab* and AB are equal, by art. 610,

610, and gk is to GK as ab or DE to AB. Any triangle mnr in the plane of the circle is to MNR its projection in the plane of the ellipse in the same ratio; for if nq parallel to ab meet mr in q, and NQ parallel to AB meet MR in Q, rh and mg parallel to ab meet nh and qk perpendicular to ab in h, l, g, and k, and GHLK be the projection of $ghlk$, the triangles MNR and mnr will be the halves of the rectangles GHLK and $ghlk$; consequently mnr will be to MNR as DE to AB. It appears from this, that any figure described in the plane of the circle is to its projection in the plane of the ellipse as DE to AB; and that any equal figures described in the former are projected by equal figures in the latter. Thus the squares described about the circle being always equal, the parallelograms described about any conjugate diameters of the ellipse (which are the projections of those squares) are always equal. If CP and CS be taken upon any diameter FI, from C in any given ratio to CF, the area of the parallelogram contained by the tangents drawn from P and S to the ellipse will be given; and thus the property of the ellipse described in the Introduction, p. 8, is easily demonstrated.

618. If the point r (*fig.* 274) describe the circumference of the circle $adbe$ with an uniform motion, and R be always the projection of r, the ray CR will describe equal areas about C in equal times, and rv and RV being arches described in the same time, and vt the subtense of the angle of contact in the circle parallel to cr being to VT the subtense of the angle of contact in the ellipse parallel to CR as ca to CR, it follows that the force directed towards the centre C by which the ellipse could be described, is to the force by which the circle arb upon the diameter ab could be described uniformly in the same time as CR to CD. The velocity in the ellipse at R is to the velocity in the circle as CZ the semidiameter conjugate to CR to CD by art. 610. And when the force towards the centre is as the distance, the periodic times in circles being equal, the times in which ellipses are described are likewise equal. Thus *prop.* X, *lib.* 1, *Princip.* with its corollaries are briefly demonstrated. In like manner, f being any point in the plane of the circle, let r move in the circumference, so that rf may describe equal areas in equal times about f; then if S and R be the projections in the plane of the

ellipse

ellipse of f and r, RS will describe equal areas in equal times about S, by art. 617, and the force at R towards S will be to the force at r towards f as SR to fr or (cg and CG parallel to the respective tangents at r and R, being supposed to meet fr and SR in g and G, rf and rc being produced till they meet the circle in x and o) as GR to gr. But fp being perpendicular to the tangent at r in p, the force at r towards f is inversely as $fp^2 \times rx$; consequently the force at R towards S is directly as GR, and inversely as $fp^2 \times grx$, or (because the rectangle grx is equal to $2cr^2$, and therefore invariable) as $\frac{GR}{fp^2}$, or (because fp is to rc as fr to gr, or as SR to GR) as $\frac{GR^3}{SR^2}$; and when S is the *focus* of the ellipse, GR being always equal to CA, the force at R towards S is inversely as the square of SR the distance from the *focus*, as was shown in art. 446.

619. Let Aa and Bb (*fig.* 275) be any two diameters of the ellipse that are perpendicular to each other, and CL the perpendicular from C on the chord AB is always of the same length. For ABab is a *rhombus*, Kl and GH that bisect AB and Ba in P and V are conjugate diameters, CG^2-CV^2 is to BV^2, or CV^2, as CG^2 to CK^2, and CG^2 to CP^2 as CG^2+CK^2 to CK^2. But KQ being perpendicular to GH in Q, CP^2 is to CK^2 as CL^2 to KQ^2; consequently CG^2 is to CG^2+CK^2 as CL^2 to KQ^2; therefore CG^2+CK^2 being invariable, and $CG \times KQ$ being likewise invariable, by art 616 and 617, it follows that CL is invariable and always equal to the perpendicular from C on the chord that joins the extremities of the transverse and second axis. Hence the area of a *rhombus* ABab inscribed in the ellipse is as AB the side of the figure, and is least when the figure is rectangular, or when Kl and GH are the axis of the ellipse, and is greatest when AB joins the extremities of the transverse and second axis.

620. Upon AB (*fig.* 276) any diameter of an ellipse take the points G and F, so that the square of GF may be equal to the rectangle AFB; from G draw a right line GE that meets the ellipse in H and K, and FE (parallel to the tangent at B) in E, then HE, GE, and KE will be in continued proportion. For let a, b, g, f, e, h, and k be the points in the plane of the circle from which

A, B,

A, B, G, F, E, H, and K are projected on the plane of the ellipse; then fe will be perpendicular to cf, and the rectangle afb, or $cf^2 - cb^2$, equal to gf^2; consequently ge^2 is equal to $ef^2 + cf^2 - cb^2$, or $ce^2 - cb^2$, that is, to the square of the tangent et, or to the rectangle hek; therefore, he, ge, and ke being in continued proportion, HE, GE, and KE are likewise proportional. It appears, in the same manner, that when G is any other point upon the diameter AB, the difference of the rectangle HEK and of the square of EG is to the square of the semi-diameter parallel to EG, as the difference of AFB and GF^2 to CB^2.

621. Let any quadrilateral figure $aefb$ (*fig.* 277) be inscribed in the circle, and gm any parallel to ef, one of its sides, meet the other sides ab, ae, bf in g, k, l, respectively, and the circle in h and m; then gh, gk, gl, and gm will be proportional; for the angle gak being equal to efb, or glb, it follows that the triangles gak and glb are similar, and the rectangle kgl equal to agb, or hgm. From this it follows, that if AEFB be any quadrilateral figure inscribed in the ellipse, and any right line GM parallel to one of the sides EF meet the other sides AB, AE, BF in G, K, L, and the ellipse in H and M, then GH, GK, GL, and GM will be proportional. In this manner, many other properties of the ellipse are briefly deduced, as *lemma* 24, 25, *lib.* 1, *Princip.* But we shall only subjoin an instance or two of the properties of the conic sections, that are briefly demonstrated, by showing first that they take place in the circle, and then transferring them to any conic section in general, by considering it as the projection of a circle upon an oblique plane, by rays that issue from a given point.

622. Let g (*fig.* 278) be a given point in the plane of the circle, ef a right line through g that meets the circle in e and f, et and ft the tangents at e and f, and their intersection t will be always found in a right line given in position; for, join cg, and let td be perpendicular to cg in d, join ct, and it will bisect ef in m. The rectangle mct is equal to ce^2, and the triangles cgm, ctd being similar, the rectangle gcd is equal to mct, and consequently to ce^2; therefore cd is given, and dt is given in position. But it is obvious, that if this figure be projected upon oblique plane by rays issuing from a given point V, the projection

tion of the circle will be a conic section, the right lines *et* and *ft* will be projected by the tangents of the conic section, the point *g* by a given point, and *td* by a right line in the plane of the conic section given in position. Therefore when G is a given point in the plane of any conic section, and EF always passes through G, the intersection of ET and FT the tangents at E and F will be always found in a right line TD given in position.

623. Let the five points C, S, E, A, and B (*fig.* 279) be in the circumference of the circle; produce BC and ES till they meet in D; let CP and SP be drawn from C and S to any point P in the circle; let CP meet EA in N, and SP meet BA in Q, then D, Q, and N will be always in a right line. For let N*n* parallel to SE meet SQ in *n*, AP in *m*, and AB in *r*; let A*n* meet the circle in *b* and SE in G, and BA meet with SE in K. Because the angle AN*n* is equal to AEK, or APS, a circle will pass through A, N, *n*, and P, and the angle NA*n* (or EA*b*) will be equal to NP*n*, or CPS; consequently the arch E*b* will be equal to CS, and C*b* parallel to SE. Let BA, BS, and AE meet C*b* in *f*, *e*, and *l*, and *fb* will be to *fl* as *fe* is to *f*C, by art. 621. But KG is to KE as *fb* to *fl*, and, because B*e*S is a right line, KS is to KD as *fe* to *f*C. Therefore KG is to KE as KS to KD; and KG being to KE as *rn* to *r*N, KS is to KD as *rn* to *r*N; and because S, Q, and *n* are in a right line, it follows that D, Q, and N are likewise in a right line. From which it follows, by supposing this figure to be projected as in the last article, that if C, S, E, B, and A be five points in a conic section, and any two of the right lines BC and ES intersect each other in D, CP and SP be drawn to any point P in the section, CP meet EA in N, and SP meet AB in Q, then D, Q, and N will be always in a right line. Hence it appears that a conic section can be drawn through those five points C, S, E, B, and A, by drawing any line DQN from D meeting AE in N and AB in Q, joining SQ and CN; for their intersection P will be a point in the conic section. And this is the method of describing a conic section through any five given points (when no more than two of those points are in a right line), that was mentioned in art. 322. The way of drawing a tangent to any point C of the conic section,

Fig. 270.

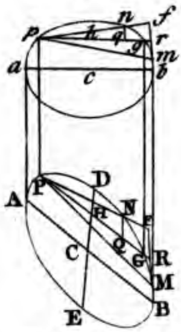


Fig. 271.

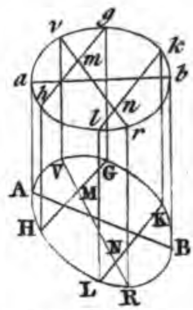


Fig. 274.

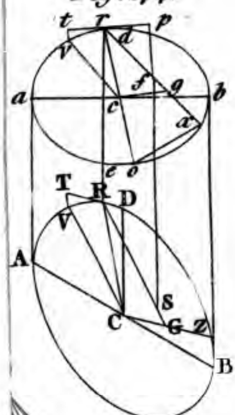


Fig. 275.

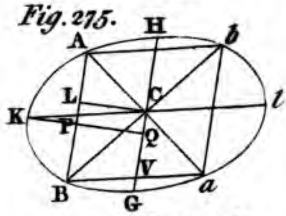


Fig. 276.

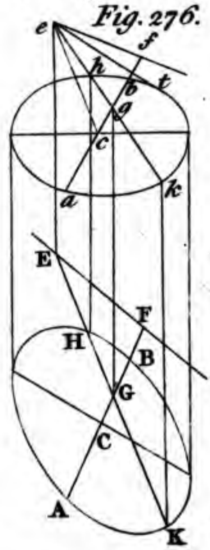
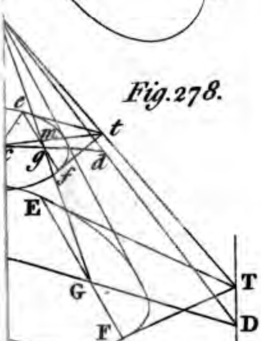


Fig. 278.



tion, that was described in art. 324, may be demonstrated in the same manner, by showing first that it takes place in the circle. By supposing one or more of the right lines that were inscribed in the conic section to be tangents, several properties of those figures may be briefly deduced from this proposition; particularly that which was mentioned (as analogous to a property of the lines of the third order) in art. 401, is the case when the right lines ESD and BCD are tangents at S and C.

624. Let P, H, and K (*fig.* 280) be any three points in an ellipse, let PM parallel to HK and KN parallel to PH meet the ellipse in M and N, and a right line through H parallel to MN will be the tangent at H. When the figure is a circle, the arch HM is equal to HN, MN is perpendicular to the diameter through H, and consequently parallel to the tangent at H. This property is extended to the ellipse, by art. 611 and 612, and may be demonstrated of any conic section. But we proceed now to these properties of the ellipse, which we had chiefly in view, because of their use in the following enquiries.

625. Let PH (*fig.* 281) be a chord of an ellipse parallel to the axis DE, LK any ordinate to this axis at V, meeting the ellipse in L and K, join DL and DK, let PM and PN parallel to DL and DK meet it in M and N, let MQ and NR be perpendicular to PH in Q and R, then the sum or difference of PQ and PR (according as Q and R are on the same or different sides of P) will be to 2DV as the chord PH to the axis DE. For supposing, first, the figure to be a circle, let the semidiameter HC meet the circumference again in I; and the arches HM and HN being equal to EL and EK, and consequently equal to one another, the right line MN will be bisected perpendicularly by the diameter HI in X. Because the arch MH is equal to LE, IX is equal to DV; let XZ be perpendicular to PH in Z, and because the angle HPI is right, PZ will be to IX (or DV) as PH to IH or DE. But QR is bisected in Z; therefore PR $\mp$ PQ is equal to 2PZ, and is to 2DV as PH to DE. This is extended to the ellipse by art. 611 and 614, and may be demonstrated of any conic section.

626. Let P*d* and H*e* (*fig.* 282) be perpendicular to the axis DE in *d* and *e*, describe an ellipse *adbe* upon the axis *de* similar

lar to ADBE. Let lk an ordinate from the internal ellipse to the axis de in v meet this ellipse in l and k. Let PM and PN parallel to dl and dk meet the external ellipse in M and N, MQ and NR be perpendicular to PH in Q and R, and PR $\mp$ PQ will be equal to $2dv$. For dv will be to DV as de (or PH) to DE, and therefore as PR $\mp$ PQ to 2DV, by the last article; consequently PR $\mp$ PQ is equal to $2dv$. This may be demonstrated in the usual manner from the property of the ellipse described at the end of art. 612.

627. Let the right line PS perpendicular to the ellipse in P meet the axis DE in S, and SZ be perpendicular to the semidiameter CP in Z, then the rectangle CPZ will be equal to the square of the semiaxis CA, that is conjugate to CD. For let PY parallel to AC meet CD in d and CO the semidiameter conjugate to CP in Y; and let PS meet CO in T. Then, because PS is to PZ as PC to PT, the rectangle CPZ is equal to SPT, which (because PT is to PY as Pd to PS) is equal to the rectangle dPY, and therefore is equal to CA^2, by art. 612. In the same manner if PS meet the axis AB in f, the rectangle fPT will be equal to CD^2, and PS will be to Pf as CA^2 to CD^2. Because dC is to dS as Pf to PS, dS is to dC as CA^2 to CD^2.

628. Supposing that the gravitation towards any particle decreases in the same proportion that the square of the distance from it increases, let PAEa, PBFb (*fig.* 283) be similar cones consisting of such particles, terminated by spherical bases AEa, BFb that have their centre in P; and the gravitation at P towards the solid PAEa will be to the gravitation at P towards PBFb as PA to PB, or in the same ratio as any homologous sides of these similar solids. For let MNm be any surface similar to AEa having its centre likewise in P; and the gravitation towards the surface AEa will be to that towards MNm in the ratio compounded of the direct ratio of the surface AEa to MNm (or PA^2 to PM^2) and of the inverse ratio of PA^2 to PM^2, that is, in a ratio of equality; consequently, the gravitation towards the surface AEaA being represented by A, the gravitation towards the solid PAEa will be represented by A × PA, and that towards the similar solid PBFb by A × PB, which are in the ratio of PA to PB. In the same manner the gravitation towards the frustum that is bounded by the surfaces AEA, MNm is represented by A × AM. It is manifest that

that though the surfaces AE*a* and MN*m* be of any other form, yet the ultimate ratio of the gravitations at P towards the conical or pyramidal solids PAE*a*, PMN*m* is that of PA to PM; and that if AQ and M*q* be perpendicular to PH in Q and *q*, these forces reduced to the direction PH will be ultimately in the ratio of PQ to P*q*.

629. The forces with which particles similarly situated with respect to similar homogeneous solids gravitate towards these solids are as their distances from any points similarly situated in the solids, or as any of their homologous sides. For such solids may be conceived to be resolved into similar cones, or frustums of cones, that have always their vertex in the particles; and the gravitation towards these cones, or frustums, will be always in the same ratio.

630 (*fig.* 282, N. 2). A particle placed within the hollow solid that is generated by the the annular space terminated by two concentric circles, or similar concentric ellipses, ADBE and *adbe*, revolving about the axis AB, has no gravity towards this solid. For let *p* be any such particle, *pk* any right line from *p* that meets the internal circle or ellipse in any points *f* and *q*, and the external figure in *x* and *r*; then if *xr* be bisected in *z*, *fq* will be likewise bisected in *z*, because the figures are similar and similarly situated; consequently *fx* is equal to *qr*; and the gravitations of *p* towards opposite frustums of the solid that have their vertex in *p* and are terminated by the same right lines produced from *p* with opposite directions will be always equal, by art. 628, and mutually destroy each other's effect.

631. It follows from this that the gravity at any point p in the semidiameter CP towards the sphere or spheroid is to the gravity at P as Cp to CP; because the gravitation towards the solid generated by the annular space that is included betwixt APB, *apb* has no effect upon a particle at p; so that the gravity at p towards the whole solid ADBE is equal to the gravity at p towards the solid *adbe*, which is to the gravity at P towards the solid ADBE as Cp to CP, by art. 629.

632. Let two planes PMKI and PNLI (*fig.* 284) intersecting each other in the right line PI include a proportion of a solid between them, that consists of particles which attract in this manner; let

let P and G be any two points in the right line PI, GK be always parallel to PM, and the planes PMN, GKL perpendicular to PMKI intersect the plane PNLI in the right lines PN and GL; then the gravitation of P towards the pyramidal solid PM*mn*N generated by the plane PMN revolving about P will be to the gravitation of G towards the pyramidal solid GK*kl*L generated by GKL revolving about G ultimately in the ratio of PM to GK, when the inclination of the planes and the equal angles MP*m*, KG*k* are supposed to be diminished till they vanish. For the right lines PM and GK being always parallel, MN will be ultimately to KL as PM to GK, and the angle MPN equal to KGL; consequently, the angles MP*m* and KG*k* being equal, the gravitation of P towards the solid PM*mn*N will be to the gravitation of G towards GK*kl*L as PM to GK, by art. 628.

633. Any sections of a spheroid made by parallel planes are similar ellipses. Let AB (*fig.* 285) be the axis of the solid, GPH a section of the spheroid by a plane perpendicular to the generating ellipse ADBE in GH and PM be perpendicular to GH in M, let KML perpendicular to AB the axis of the solid meet the ellipse ADBE in K and L, and CQ be the semidiameter parallel to GH. Then because PM is perpendicular to the plane ADBE, the points K, P, and L will be in a semicircle described upon the diameter KL, and the square of PM equal to the rectangle KML, which (by art. 615) is to the rectangle GMH as CD^2 to CQ^2; consequently the section GPH is an ellipse, and is similar to any other section of the solid by a parallel plane, the ratio of the axis GH to the other axis being that of CQ to CD. From this it follows that the sections of two similar concentric spheroids similarly situated, which are made by the same plane, are similar ellipses; because they are similar to the sections of the same solids by a parallel plane that passes through their common centre; and these last are similar by art. 122. It appears likewise that all sections of the spheroid made by planes perpendicular to the circle generated by the axis CD (which we may call the *Equator* of the solid) are similar to the generating ellipse ADBE.

634. The gravity of any particle of a sphere or spheroid being resolved into two forces, one perpendicular to the axis of the

the solid, the other perpendicular to the plane of its equator, all particles equally distant from the axis tend towards it with equal forces, and all particles at equal distances from the plane of the equator gravitate equally towards this plane, whether the particles be at the surface of the solid or within it. And the forces with which particles at different distances from the axis tend towards it are as these distances: the same is to be said of the forces with which they tend towards the plane of the equator. This easily appears of the sphere from what was shown in art. 631, and is mentioned for the sake of the analogy only. Let P (*fig.* 286) be any point in the surface of a spheroid, APDBE a section of the solid through its axis AB, P*f* a perpendicular to AB in *f*, P*d* a perpendicular to the equator of the solid in *d*; and the gravity at P towards the solid being resolved into a force in the direction P*f* and another force in the direction P*d*, the former will be equal to the gravity at *d* towards the solid, and the latter equal to the gravity at *f*. Let *adbe* be a spheroid similar to ADBE having the same centre C and its axis *ab* in the same right line AB with the axis of the external solid. The sections of these spheroids by any plane that passes through the right line P*d*I will be similar concentric ellipses similarly situated by art. 633, and the gravity of P in the direction P*f* perpendicular to the axis AB that arises from the attraction of any portion or slice of the external solid contained by two such planes will be equal to the gravity at *d* in the direction *d*C which arises from the attraction of the slice of the internal solid that is contained by the same planes. To demonstrate this, let PMNIG (*fig.* 287), P*mn*I*g* (in the next figure) be the sections of the external solid by two such planes, *d*KL*d* and *dkld* the sections of the internal solid by the same planes; let KL be an ordinate at V to *de*, the axis of the internal ellipse which is in the plane of the equator of the solid, join *d*K and *d*L, and let PM and PN be always parallel to *d*K and *d*L, respectively. Let the planes PM*m*, PN*n*, *d*K*k*, *d*L*l* perpendicular to the plane PMNIG meet P*mn*I*g* in the right lines P*m*, P*n*, *dk*, and *dl*, respectively; and let those planes revolve about the points P and D, PM being always parallel to *d*K and PN to *d*L, while V is supposed to describe the right line *ed*. Then the forces with which P and *d* are attracted towards the

the pyramidal solids generated by the planes PM*m*, PN*n*, *d*K*k*, and *d*L*l* will be ultimately as the right lines PM, PN, *d*K, and *d*L, respectively, by art. 632; and Pp being parallel to *de*, and MQ and NR perpendicular to Pp in Q and R, if these forces be resolved into such as act in the directions Pp and *de*, and such as act in the right lines perpendicular to these, the former will be as the right lines PQ, PR, *d*V, and *d*V. But PR+PQ is always equal to 2*d*V by art. 626. Therefore the gravity of P in the direction Pp arising from the attraction of the pyramidal solids generated by the planes PM*m* and PN*n* is ultimately equal to the gravity of *d* in the direction *de* arising from the attraction of the pyramidal solids generated by the planes *d*K*k* and *d*L*l*. And since this always holds, while we conceive the point V to describe *ed*, and these planes to describe the portions of the external and internal solids terminated by PMNIG and P*mn*I*g*, it follows that the gravity of P in the direction Pp arising from the attraction of the whole portion of the external solid bounded by the planes PMNIG, P*mn*I*g* is ultimately equal to the gravity of *d* in the direction *de* that arises from the attraction of the part of the internal solid bounded by the same planes, when the angle contained by those planes is supposed to be diminished till it vanish. By (*fig.* 286) conceiving other slices of the solids contained by planes that pass through P*d*I, and form equal angles with the plane APDB on the other side, to attract the particles P and *d*, it will appear that the gravities of P and *d* towards the axis of the spheroid arising from the joint attraction of those slices will be equal. And since this holds of all the portions of the solids contained by such planes, it follows that the force with which P tends towards the axis AB arising from the attraction of the whole spheroid, is equal to the gravity of *d* towards the internal solid, or (by art. 631) to its gravity towards the whole external solid ADBE. The gravity of any particle *p* situated in the right line P*d* towards the spheriod ADBE, is equal to its gravity towards a similar concentric spheroid similarly situated that has C*p* for its semidiameter, by art. 631, and therefore its gravity in the direction perpendicular to AB is equal to the gravity of *d* towards the solid *adbe*, by what has been shown. Therefore all particles equally distant from the axis tend towards it with equal forces;

and

and because the gravity at d is to the gravity at D as Cd to CD, by art. 631, it follows that the gravity of P towards the axis is to the gravity at D towards the spheroid as Pf to DC. In the same manner it is shown, that the gravity of P towards the plane of the equator is equal to the gravity of f towards the spheroid ADBE, and is to the gravity at A towards the spheroid as fC or Pd to AC.

635. In order therefore to find the direction in which the spheroid attracts any particle at P, and the force of this attraction, let A denote the attraction at the pole A, and D the attraction at the equator, let Pd be perpendicular to the plane of the equator in d, upon dC take dQ from d towards C, so that dQ may be to dC as D × CA to A × CD, join PQ; the attraction towards *the spheroid* will tend in the direction PQ, and be always measured by this right line PQ. For the gravity towards the spheroid in the direction Pd being to A the gravity at the pole as Pd to AC, and the gravity at P in the direction Pf, or dC, being to D as dC to DC, by the last article; it follows, that the gravity at P in the direction Pd is to the gravity at P in the direction Pf as $A \times \frac{Pd}{AC}$ to $D \times \frac{dC}{DC}$, that is (by the supposition) as Pd × dC to dQ × dC or as Pd to dQ; consequently the gravity at P is in the direction PQ; and if the gravity at A towards the spheroid be represented by AC, the gravity at P in the direction Pd will be represented by Pd, and the gravity at P towards the spheroid by PQ. In the same manner if fq be taken upon the axis from f towards C in the same ratio to fC as A × CD to D × CA, then Pq will always show the direction and measure the force of the gravity at any point P towards the spheroid, supposing the gravity at D to be represented by DC. Let Dx perpendicular to CD represent the gravity at D, join Cx, and besause the gravity of any particle in the semidiameter CD is as its distance from C, the gravity of the column DC (the spheroid being supposed to be fluid) will be measured by the triangle CDx or ½ CD × Dx, or ½ CD × D. In the same manner the gravity of the column AC will be measured by ½ AC × A. And as the columns CD and AC gravitate equally in the sphere, so the gravity of the column CD is

 greater

greater or less than the gravity of the column AC in the spheroid according as it is oblate or oblong, and CD × D is greater or less than CA × A in the spheroid, according as CD is greater or less than CA (because the fluid would sink in the former case at D, and in the latter at A, till its figure becomes spherical). But this will appear more fully afterwards when we come to determine the ratio of A to D in a given spheroid.

636. Hitherto we have supposed the particles of the spheroid to be affected only by their mutual gravitation towards each other. Let us now suppose any new powers to act upon all the particles of the spheroid in right lines, either perpendicular to the axis of the spheroid, or to the plane of its equator; or some powers to act in right lines perpendicular to the axis, and others in lines parallel to it; and let each force vary always as the distance of the particles from the axis, or equator, to which the direction of the force is supposed perpendicular. Then the spheroid being supposed to be fluid, if CA be to CD inversely as the whole forces that act on equal particles at A and D, the fluid will be every where *in æquilibrio*. To demonstrate this proposition fully, we shall show, 1. That the force which results from the attraction of the spheroid and those extraneous powers compounded together acts always in a right line perpendicular to the surface of the spheroid. 2. That the columns of the fluid sustain or balance each other at the centre of the spheroid. And, 3. That any particle in the spheroid is impelled equally in all directions.

637. 1 (*Fig.* 286). Let the forces that result from the attraction of the spheroid and the extraneous powers at A and D be called M and N; and M will be to N as CD to CA, by the supposition. Because the attraction of the spheroid at P in the direction Pd is to its attraction at A as Pd to AC, and the force of each extraneous power at P is supposed to be to the force of the same power at A in the same ratio of Pd to AC, it follows that the whole force by which a particle at P tends in the direction Pd is to M as Pd to AC. In the same manner the whole force with which a particle at P tends in the direction Pf is to N as Pf or dC to DC; consequently the force with which P tends in the direction Pd is to the force with which it tends in the direction

Pf

Pf as $M \times \frac{Pd}{AC}$ to $N \times \frac{dC}{DC}$; and supposing PK that meets CD in K to be the direction in which a particle at P tends towards the spheroid from the composition of those two forces, Pd will be to dK in the same ratio; so that dK will be to dC as $N \times AC$ to $M \times DC$, or (because N is to M as AC to DC, by the supposition) as AC^2 to DC^2. But if PK was supposed perpendicular to the ellipse APDB at P, dK would be to dC in this same ratio, by art. 627. Therefore any particle as P at the surface of the spheroid tends towards it in a right line perpendicular to its surface; and the force M which acts on a particle at the pole A being represented by the semiaxis AC, the force which acts on an equal particle at any point of the surface P will be always represented by the perpendicular PK terminated by the plane of the equator of the solid in K. It appears likewise that any particle p within the spheroid in the semidiameter CP tends in the direction pk parallel to PK, with a force that is measured by the right line pk terminated by the same plane in k, because the forces that act on P and p in right lines perpendicular to the axis, or equator, are as the distances from the axis, or equator, by the supposition.

638. In order to show, that when the spheroid is fluid, the columns sustain each other at the centre, let KZ (*fig.* 286) and kz be perpendicular to PC in Z and z; then the force M which acts at the pole A being represented by the semiaxis AC, the force with which particles at P and p tend in the direction PC will be represented by PZ and pz respectively; and because pz is to PZ as Cp to CP, the gravity of the whole column PC in the direction PC will be measured by $\frac{1}{2}$ PZ $\times$ PC, which is equal to $\frac{1}{2}$ CA^2 (by art. 627), or to $\frac{1}{2}$ CA $\times$ M. Therefore the gravity of any column PC in the direction PC is equal to the gravity of the column AC in the direction AC, and all the columns of the fluid sustain each other at C.

639. Let p (*fig.* 288) be any particle in the spheroid, Pp a column from the surface to the point p, produce Cp till it meet the surface in q; upon CA take CO in the same ratio to CA as Cp is to Cq, and the gravity of the column Pp in the direction Pp will be equal to the gravity of the column AO in the direc-

tion AC. First, let Pp be in the plane APDB, let PG and pg be perpendicular to the plane of the equator in G and g, PL and pl perpendicular to the axis AB in L and l; and Pp being supposed to meet AB in f and DE in h, let ge and lu be perpendicular to Pp in e and u. Then the force with which the particle p tends in the right line parallel to the axis will be to M as pg to AC, by the supposition; and this force reduced to the direction Pp will be to M as pe to AC. The force with which p tends in the direction perpendicular to the axis is to N as pl to DC, and this force reduced to the direction Pp is to N as pu to DC. Therefore the whole force with which p tends in the direction Pp is $M \times \frac{pe}{AC} + N \times \frac{pu}{DC}$, or $M \times \frac{ph}{AC} \times \frac{pg^2}{ph^2} + N \times \frac{pf}{DC} \times \frac{Pl^2}{pf^2}$. From which it follows that the gravity of the whole column Pp in the direction Pp is $\frac{M}{2AC} \times \frac{pg^2}{ph^2} \times \overline{hP^2 - hp^2} + \frac{N}{2DC} \times \frac{pl^2}{pf^2} \times \overline{fP^2 - fp^2}$, that is $\frac{M}{2AC} \times \overline{PG^2 - pg^2} + \frac{N}{2DC} \times \overline{PL^2 - pl^2}$. But PL^2 is to $CA^2 - CL^2$, and pl^2 to $CO^2 - Cl^2$ as CD^2 to CA^2; consequently $PL^2 - pl^2$ is to the difference of $CA^2 - CL^2$ and $CO^2 - Cl^2$ in the same ratio, and (because M is to N as DC to AC) the whole gravity of the column Pp in the direction Pp will be to $\frac{1}{2} M \times AC$ as $CL^2 - Cl^2 + CA^2 - CO^2 + Cl^2 - CL^2$ to CA^2, that is as $CA^2 - CO^2$ is to CA^2, and consequently equal to the gravity of the column AO in the direction AC. Therefore the particle p is pressed equally in all directions in the meridian plane APDB that passes through p. In like manner it is shown, that any other columns from the surface of the spheroid to the particle p press equally upon it, and sustain each other.

640. We conclude, therefore, that when the particles of a fluid spheroid of an uniform density gravitate towards each other, with forces that are inversely as the squares of their distances from each other, and any other powers act on the particles of the fluid, either in right lines perpendicular to the axis

N°. 2.

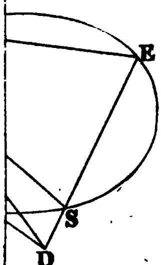


Fig. 281.

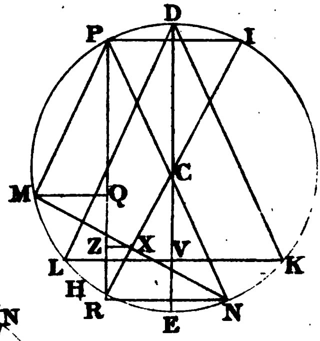


Fig. 280.

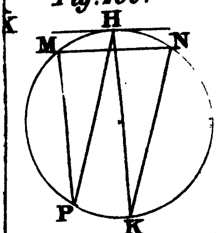


Fig. 284.

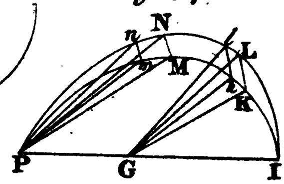
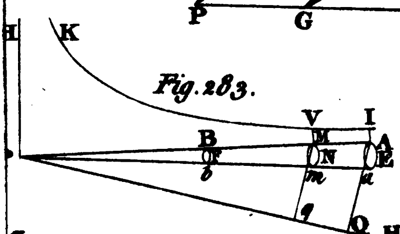


Fig. 283.



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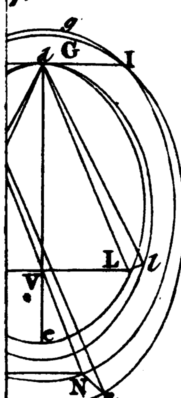
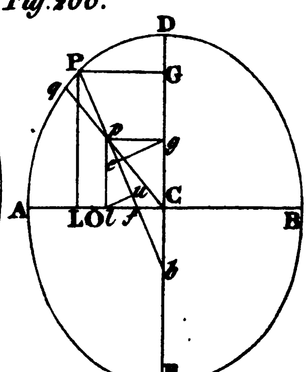


Fig. 288.



axis that vary in the same proportion as the distances from the axis, or in right lines perpendicular to the plane of the equator that vary as their distances from it, or when any powers act on the particles of the spheroid that may be resolved into such as these; then if the whole force that acts at the pole A be to the whole force that acts at the circumference of the equator as the radius of the equator to the semiaxis of the spheroid, the fluid will be every where *in æquilibrio*; surfaces similar and concentric to ADBE will be the level surfaces at all depths; and the forces with which equal particles at those surfaces tend towards the spheroid, will be measured by perpendiculars to the surfaces terminated either by the plane of the equator, or by the axis of the spheroid.

641. This theorem is of use in several philosophical enquiries. Suppose first a fluid spheroid ADBE (*fig.* 286) of an uniform density to revolve on its axis AB; let the attraction of the spheroid at the pole A be represented by A, the attraction at the circumference of the equator by D, the centrifugal force there arising from the rotation of the spheroid by V; then if CA be to CD as D—V to A, or V be equal to the excess of D above A $\times \frac{CA}{CD}$, the fluid will be every where in *æquilibrio*. For in this case M is equal to A, there being no centrifugal force at the pole; the force N that acts on any particle in the circumference of the equator is equal to D—V the excess of the attraction above the centrifugal force there; and the centrifugal force, with which any particle of the spheroid endeavours to recede from its axis in consequence of the rotation of the spheroid, is as its distance from the axis; consequently if A be to D — V as CD to CA, the fluid will be *in æquilibrio* in all its parts, by what has been shown. It appears therefore that if the earth, or any other planet, was fluid, and of an uniform density, the figure which it would assume in consequence of its diurnal rotation would be accurately that of an oblate spheroid generated by an ellipsis revolving about its second axis, as Sir *Isaac Newton* supposed: and we cannot but observe, that as no theory of gravity has a foundation in nature but his only, so no other gives so simple a figure of the planets, as will appear

by comparing what was demonstrated above in art. 492. This theorem is applicable in like manner to the theory of the tides. But before we proceed to a more particular application of it, we are first to show how the gravity towards a spheroid at the pole is easily measured by a circular or hyperbolic area, according as the spheroid is oblate or oblong; and how the gravity towards it at the circumference of the equator, or at any distance in the axis, or in the plane of the equator produced, is determined from the gravity at the pole, without any new quadrature or computation. For this end we premise the following *lemma*.

642. Let AD*da* (*fig*.289) be any section of a solid of an uniform density by a plane that passes through a given point P, and PC, PH be right lines given in position in this plane; let any right line PM drawn from P meet the figure AD*da* in M and *m*, and a circle BN*b* described with the given radius PC in N; let MQ and *mq* be always perpendicular to PC in Q and *q*, and NR be perpendicular to PH in R; upon RN take RK equal to PQ — P*q*; and let the ordinate RK always determined in this manner generate the area HG*gh*, while PM revolves about P from PA to P*a*. Then if we suppose another plane that passes through the right line PH to cut the same solid, the gravity of the particle P towards the slice of the solid included betwixt those two planes and that stands upon the base AD*da*, when reduced to the direction PC, will be ultimately as $\frac{HGgh}{PC}$, the angle contained by the two planes being supposed to be continually diminished till it vanish. For let another right line PS meet the figure AD*da* in S and *s*, and the circle BN*b* in *n*; let MZ and *nr* be perpendicular to PH in Z and *r*, the arch M*o* described from the centre P meet PS in *o*, and M*x*, *ou* be perpendicular on the other plane that is supposed to pass through PH in *x* and *u*. Then if the point P be without the figure AD*da*, the gravity at P towards the pyramidal solid PM*oux* will be ultimately as M*m* × $\frac{Mo \times Mx}{PM^2}$, by art. 628, or (because M*x* is as MZ) as M*m* × $\frac{Mo \times MZ}{PM^2}$, that is (because M*o* is to N*n* as MZ to

to NR, and the rectangle $Mo \times MZ$ to $Nn \times NR$ as PM^2 to PC^2) as $Mm \times \frac{Nn \times NR}{PC^2}$, or ($Nn$ being to Rr as PC to NR) as $Mm \times \frac{Rr}{PC}$; and the gravity of P towards that solid reduced to the direction PC, will be as $Qq \times \frac{Rr}{PC}$ or $\frac{RK \times Rr}{PC}$. But $RK \times Rr$ ultimately measures the fluxion of the area HGKR. Therefore the gravity of P in the direction PC, that arises from the attraction of the whole slice of the solid which has the figure AD*da* for its base, is ultimately as $\frac{HGgh}{PC}$, the angle contained by the planes which terminate the slice being continually diminished. When the point P is betwixt M and *m*, then RK is to be taken equal to PQ—P*q*, and the gravity at P is measured in the same manner. It follows from this lemma, that, supposing the figure AD*da* to revolve about the axis PH, and to generate a solid, and the direction PC to coincide with PH, the gravity at P towards this whole solid will be as $\frac{HGgh}{PC}$. When the particle P is so situated with respect to the figure AD*da*, that the perpendiculars from the points M intersect PC on different sides of P, the gravity at P in the direction PC is to be determined from the difference of the areas generated by the ordinate RK.

643. Let a particle at P (*fig.* 290) gravitate towards the sphere generated by the semicircle ADB about the axis AB, and C being the centre of the sphere, let any right line PM meet the semicircle in M and *m*, and the circle CNH described from the centre P in N; let NR be perpendicular to PC in R, and RK be always equal to Q*q* when P is without the sphere or in contact with it. Let CL be perpendicular to PM in L, and M*m* being bisected in L, LM^2 will be equal to $PL^2 - MPm$ or $PR^2 - APB$, and the fluxion of LM^2 equal to the fluxion of PR^2; so that the fluxion of PR will be to the fluxion of LM as LM to PR. And because KR or Q*q* is to 2LM as PR to PN, the fluxion of the area generated by the ordinate KR is in this case

equal

equal to the rectangle contained by $\frac{2LM^2}{PC}$ and the fluxion of LM. Therefore the area IRK is equal to $\frac{2LM^3}{3PC}$, and the gravity at P towards the portion of the sphere generated by the segment MDm about the axis AB is as $\frac{2LM^3}{3PC^2}$, and consequently as the cube of Mm the chord of the segment MDm, directly, and the square of PC the distance of the particle from the centre inversely. Hence the gravity at P towards the whole sphere is as the cube of its diameter (or its quantity of matter, the density being given), directly, and the square of PC inversely; and is the same as if we should conceive the whole matter in the sphere to be collected in its centre. The same is to be said of the gravity towards the aggregate of any number of such spheres that have a common centre; from which it follows, that however variable the density of a sphere may be at different distances from the centre, providing the densisty be always the same at the same distance from it, the gravity of a particle (that is not within the sphere) towards it will be as the quantity of matter in the sphere directly, and the square of the distance of the particle from its centre inversely. It appears from what has been shown, that the whole area IKGC is equal to $\frac{2AC^3}{3PC}$, and that the gravity at A towards the sphere ADBE is measured by $\frac{2AC}{3}$ according to the last article.

644. Let ADBE (*fig.* 291, *N.* 1 and 2) be now a spheroid of an uniform density, generated by the semi-ellipse ADB revolving about the axis AB; let AM any right line from the pole A meet the ellipse in M and the circle CNH in N; let MQ and NR be perpendicular to AB in Q and R; upon RN take RK always equal to AQ, and let this ordinate RK generate the area AKGC while AM revolves about A and describes the area of the semi-ellipse ADB. Then the gravity at the pole A towards the spheroid ADBE will be measured by $\frac{AKGC}{AC}$ (by art. 642), and is to the gravity at A towards a sphere described upon the diameter AB

(which

(which is measured by $\frac{2}{3}$ AC, by the last article) as the area AKGC to $\frac{2}{3}$ CA².

645. The gravity at D at the circumference of the equator towards the spheroid is to the gravity at D towards a sphere described upon the diameter of the equator as $2CA^2 - AKGC$ to $\frac{4}{3}$ CA^2, and to the gravity at the pole A as $2CA^2 - AKGC$ to $2AKGC \times \frac{CD}{CA}$. For suppose the two elliptic sections DBEA and D*bea* to be perpendicular to the plane of the equator of the solid, and to intersect each other in the right line *hdg* their common tangent at D; let any right line D*m* from D meet the ellipse in *m*, and the circle *cnh* described from the centre D with the radius D*c* (equal to AC) in *n*, let *mq* be perpendicular to DE in *q*, and *nr* perpendicular to D*h* in *r* meet *mq* in *k*; and let *hk*ED be the area generated by the ordinate *rk*, while D*m* revolves about D and describes the elliptic area DBE; then the gravity at D towards the slice of the spheroid contained by the planes DBEA and D*bea* will be ultimately measured by $\frac{hkED}{Dc}$, the angle contained by those planes being given, by art. 642. But if RK produced meet GI parallel to AC in *x*, and the right lines AM and D*m* revolve about A and D so that the angle *h*D*m* be always equal to BAM; then D*n* being equal to AN, and the angle *r*D*n* to RAN, D*r* will be always equal to AR, and *hr* equal to CR or G*x*. Because qm^2 is to the rectangle D*q*E as the rectangle AQB to QM², and the triangles D*qm*, MQA being similar, *qm* is to D*q* as AQ to QM, it follows that *qm* is to *q*E as QB to QM, and D*q* to *q*E as QB to AQ; consequently DE and BA are divided in the same proportion in *q* and Q, so that D*q* is to QB, or *rk* to *x*K, as DE to AB. Therefore the bases *hr* and G*x* being always equal, the area *hrk* is to the area G*x*K in the same constant ratio of DE to AB, and the area *hk*ED to GKAI (or $CA \times AB - AKGC$) as CD to CA. When ADBE is supposed to be a circle described upon the diameter DE, or CD is supposed equal to CA, the area AKGC is equal to $\frac{2}{3}$ CD² (by art. 643), and GKAI equal to $\frac{4}{3}$ CD². Therefore by article 642, the gravity at D towards the slice of the spheroid contained by the planes DBEA and

and *Dbea* is to the gravity at D towards the slice of the sphere described upon the diameter DE that is contained by the same planes, as $\frac{hIED}{Dc}$ to $\frac{4CD}{3}$, that is, as $\frac{2CA^2 - AKGC}{CA} \times \frac{CD}{CA}$ to $\frac{4}{3}$ CD, or as $2CA^2 - AKGC$ to $\frac{4CA^2}{3}$. The gravity at D towards the spheroid is to the gravity there towards the sphere described upon the diameter DE in the same ratio; because the section of the spheroid by any plane perpendicular to the equator is always an ellipse similar to DBEA, and the section of the sphere described upon the diameter of the equator made by the same plane is always a circle having that axis of the former which is homologous to DE for its diameter; and the gravity at D towards the elliptic slice of the spheroid contained by any two such planes, is always ultimately in the same ratio to the gravity at D towards the circular slice of the sphere contained by the same planes. Therefore the gravity at D towards the spheroid ADBE is to the gravity at D towards the sphere described upon the diameter of the equator as $2CA^2 - AKGC$ to $\frac{4}{3}$ CA^2. But the gravity at D towards this sphere is to the gravity at A towards a sphere described upon the axis AB as CD to CA; and this latter gravity is to the gravity at A towards the spheriod ADBE as $\frac{2}{3}CA^2$ to AKGC by the last article; consequently the gravity at D towards the spheroid is to the gravity at A towards it as $2CA^2 - AKGC$ to $2AKGC \times \frac{CA}{CD}$. It appears likewise that the gravity at A towards a sphere, described upon the axis AB being represented by $\frac{2}{3}$ CA according to article 643, the gravity at A towards the spheroid will be measured by $\frac{AKGC}{AC}$, and the gravity at D towards it by $\frac{2CA^2 - AKGC}{2CA^2} \times CD$ or $CD - \frac{AKGC}{2CA} \times \frac{CD}{CA}$.

646. In order to measure the area AKGC, let F be the focus of the generating ellipse, and because AQ is to QM (or the rectangle $AQ \times QM$ to QM^2) as AR to RN, and QM^2 is to $AQ \times QB$ as CD^2 to CA^2, it follows that $AQ \times QM$ is to $AQ \times$

QB

QB (or QM to QB) as AR $\times$ CD^2 to RN $\times$ CA^2; consequently AQ is to QB as $AR^2 \times CD^2$ to $RN^2 \times CA^2$, and (because RN^2 is equal to $CA^2 - AR^2$) AQ, or RK, to AB as $AR^2 \times \frac{CD^2}{CA^2}$ to $CA^2 + \frac{CF^2}{CA^2} \times AR^2$ or $CA^2 - \frac{CF^2}{CA^2} \times AR^2$, according as the spheroid is oblate or oblong; that is (if Cf be taken upon CF in the same ratio to AR as CF is to AC), as $Cf^2 \times \frac{CD^2}{CF^2} Af$ in the former case, and to $CA^2 - Cf^2$ in the latter. In the former case, let Af and AF meet the circle CNH in f and S; and the fluxion of $Cf - Cf$ will be to the fluxion of Cf as Cf^2 to Af^2 (art. 195), that is, as RK $\times$ CF^2 to AB $\times$ CD^2, and to the fluxion of AR as RK $\times$ CF^3 to $2CA^2 \times CD^2$, consequently the fluxion of the area ARK will be to the fluxion of $2CA \times \overline{Cf - Cf}$ as CA $\times$ CD^2 to CF^3, ARK will be to $2CA \times \overline{Cf - Cf}$ in the same ratio, and the whole area AKGC equal to $\frac{2CA^2 \times CD^2}{CF^3} \times \overline{CF - CS}$. Therefore the gravity at A towards the sphere described upon the axis AB being represented by $\frac{2}{3}$ AC, the gravity at A towards the spheroid ADBE will be measured by $\frac{2CA \times CD^2}{CF^3} \times \overline{CF - CS}$; the gravity at D towards the same spheroid (art. 645), by CD $- \frac{CD^3}{CF^3} \times \overline{CF - CS}$ or $\frac{CD^3 \times CS - CD \times CF \times \overline{CD^2 - CF^2}}{CF^3}$, that is by CD $\times$ $\frac{CD^2 \times CS - CA^2 \times CF}{CF^3}$; and the gravity at A to the gravity at D, as $2CA \times CD \times \overline{CF - CS}$ to $CD^2 \times CS - CA^2 \times CF$, or if the arch FO described from the centre A meet CB in O, as CD $\times$ $\overline{CF - CS}$ to the segment FCO; because this segment is equal to $\frac{1}{2}$ CD $\times$ FO $- \frac{1}{2}$ CA $\times$ CF, or to $\frac{CD^2 \times CS - CA^2 \times CF}{2CA}$.

647. When CA is greater than CD, that is when ADBE (*fig.* 291, N. 2), is an oblong spheroid, the rest remaining as in the last article, let LC be taken upon CA equal to the logarithm of the ratio of CD to AF, or of the subduplicate ratio of BF to AF, the *modulus* being AC; and the gravity at the pole A will be to the gravity

gravity at D as $2CA \times CD \times LF$ to $CA^2 \times CF - CD^2 \times CL$. For in this case we found that the ordinate RK was to AB as $Cf^2 \times \frac{CD^2}{CF^2}$ to $CA^2 - Cf^2$. But if Cl be taken upon CA, so as to represent the logarithm of the ratio of $\overline{\sqrt{CA+Cf}}$ to $\overline{\sqrt{CA-Cf}}$, the *modulus* being AC, the fluxion of $Cl - Cf$ will be to the fluxion of Cf as Cf^2 to $CA^2 - Cf^2$, or as $RK \times CF^2$ to $AB \times CD^2$, and to the fluxion of AR as $RK \times CF^3$ to $2CA^2 \times CD^2$; consequently the fluxion of the area ARK is to the fluxion of $2CA \times \overline{Cl - Cf}$ as $CA \times CD^2$ to CF^3, and the whole area AKGC is equal to $\frac{2CA^2 \times CD^2}{CF^3} \times LF$. Therefore the gravity at A towards the oblong spheroid ADBE is measured by $\frac{2CA \times CD^2}{CF^3} \times LF$, the gravity at D (art. 645), towards the same spheroid by $CD - \frac{CD^3}{CF^3} \times LF$ or $CD \times \frac{CA^2 \times CF - CD^2 \times CL}{CF^3}$; and the gravity at A to the gravity at D as $2CA \times CD \times LF$ to $CA^2 \times CF - CD^2 \times CL$. What has been shown concerning the gravity at the pole A, agrees with what was advanced long ago by Sir *Isaac Newton* and Mr. *Cotes*, who contented themselves with an approximation in determining the gravity at the equator, which is exact enough when the spheroid differs very little from a sphere. The approximations proposed lately for this purpose, *Phil. Trans.* N. 438 and 445, are more accurate; and Mr. *Stirling*, after determining the gravity at the equator by a converging series, since found that the sum of the series could be assigned from the quadrature of the circle. It was shown in art. 645, how this gravity at the equator is deduced accurately from the gravity at the pole, without any new quadrature or computation. The gravity in any other latitude is determined from what has been demonstrated by art. 635 (*fig.* 286), where dQ is to be taken in the same ratio to dC as $CD^2 \times CS - CA^2 \times CF$ to $2CD^2 \times \overline{CF - CS}$, that PQ may measure the force and show the direction of the gravity at P. The gravity at any distance in the axis of the spheroid, or the plane of the equator produced, is likewise accurately determined

determined from what has been shown by the following *lemma*, without any new computation.

648. Let ADB, Pdp (*fig.* 292) be two semi-ellipses that have the same centre C and the same focus F. Let any right line PmM from P meet the internal ellipse in m and M, and Px meet the external ellipse in x, so that CL the perpendicular from C on Px may be to CR the perpendicular from C on PM as Cd to CD, then Mm will be to Px as CA to CP. For let Pyp and AvB be semicircles described upon the diameters Pp and AB; let mv and MV parallel to CD meet the circle AvB in v and V, and xy parallel to CD meet the circle Pyp in y; produce mv and xy till they meet Pp in q and I; and let Cr and Cl be perpendicular to Pv and Py in r and l respectively. Then since Pm^2 is to PC^2 as qm^2 to CR^2, and Px^2 to PC^2 as Ix^2 to CL^2, it follows that Pm^2 is to Px^2 as $\frac{qm^2}{CR^2}$ to $\frac{Ix^2}{CL^2}$, that is, by the supposition, as $\frac{qm^2}{CD^2}$ to $\frac{Ix^2}{Cd^2}$, or (because qm^2 is to the difference of qm^2 and qv^2 as CD^2 to CF^2, and Ix^2 is to the difference of Ix^2 and Iy^2 as Cd^2 to CF^2) as the difference of qm^2 and qv^2 to the difference of Ix^2 and Iy^2; consequently Pm^2 is to $Pm^2 - qm^2 + qv^2$, or Pv^2 as Px^2 to Py^2, and Pm to Pv as Px to Py. And because qv is to qm as CA to CD, and Iy to Ix as CP to Cd, so that $\frac{qv}{CA}$ is to $\frac{Iy}{CP}$ as $\frac{qm}{CD}$ to $\frac{Ix}{Cd}$; and therefore as Pm to Px, or (by what has been demonstrated) as Pv to Py, it follows that $\frac{qv}{Pv}$ is to $\frac{Iy}{Py}$ as CA to CP: therefore Cr is to Cl as CA to CP. It follows, that the triangles Crv, ClP are similar, and Vv (or $2rv$) to Py (or $2Pl$) as CA to CP. But Mm is to Vv as Pm to Pv, or Px to Py; consequently Mm is to Px as Vv to Py, or as CA to CP. It appears from hence, that when two ellipses Pdp ADB have the same centre and focus, if any semidiameters CE and Ce of those ellipses constitute angles pCE, pCe with the axis Cp, whose sines are in the same ratio as CD to Cd, these semidiameters will be to each other as CP to CA. For if CE and

and Ce be respectively parallel to Px and Pm, CE will be to Ce as Px to Mm.

649. The ellipses Pdp, ADB that have the common centre C and focus F being supposed to revolve about the axis PCp, and to generate spheroids of the same density, the gravities at P towards these solids will be in the same ratio as the quantities of matter contained in them, or as $Cd^2 \times CP$ to $CD^2 \times CA$. For let any right line PmM from P meet the internal ellipse in m and M, and the circle CNH described from the centre P in N, and Px meet the external ellipse in x and CNH in L; let mq, MQ, xI, NR, and LZ be perpendicular to Pp in q, Q, I, R, and Z respectively; upon RN take RK always equal to Qq, and upon ZL take Zk always equal to PI, and the gravity at P towards the internal solid will be to the gravity at P towards the external solid, as the area generated by the ordinate RK to the area generated by the ordinate Zk: suppose LZ to be always to NR as Cd is to CD, then Mm will be always to Px as CA to CP, by the last article. But Qq is to Mm as PR to PC, and PI to Px as PZ to PC; and the fluxion of the area CRKG is to the fluxion of the area $CZkg$ in the compound ratio of Qq to PI, and of the fluxion of PR to the fluxion of PZ, that is, in the compound ratio of Mm to Px, and of the fluxion of PR^2 to the fluxion of PZ^2 (or of the fluxion of NR^2 to the fluxion of LZ^2), and consequently in the compound ratio of CA to CP, and of CD^2 to Cd^2; and the areas CRKG, $CZkg$ being in the same ratio, it follows that the gravity at P towards the portion of the internal spheroid that is generated by the segment $AmMB$ is to the gravity at P towards the portion of the external spheroid generated by the segment Pxp, as $CA \times CD^2$ to $CP \times Cd^2$, and that the gravities at P towards the whole spheroids are in the same ratio; because Px by revolving about P describes the semi-ellipse Pdp, while mM describes the semi-ellipse ADB.

650. Hence the gravity towards the oblate spheroid ADBE at any point P in its axis produced beyond A, is measured by $\frac{2CA \times CD^2}{CF^3} \times \overline{CF - CS}$, PF being supposed to meet the arch CNH described from the centre P in S; because the gravity

at

at P towards the external solid Pdpe is measured by $\frac{2CP \times Cd^2}{CF^3} \times \overline{CF - CS}$ (art. 646), which is to the gravity at P towards ADBE as CP $\times$ Cd^2 to CA $\times$ CD^2, by the last article. In the same manner the gravity at P towards an oblong spheroid ADBE (*fig.* 291, *N.*2) is measured by $\frac{2CA \times CD^2}{CF^3} \times$ LF, CL being the logarithm of the ratio of Cd to PF, the *modulus* being PC (*fig.* 292). Because the gravity at P towards any spheroid ADBE that has its centre in C and *focus* in F, and is described on any axis AB that is not greater than Pp, is as the quantity of matter in that spheroid, it follows that if the density of the solid Pdpe vary, but so as to be always the same over the surface of any such spheroid ADBE, the gravity towards Pdpe in this case will be to the gravity towards it when its density is uniform, as the quantity of matter contained in it in the former case to the quantity of matter contained in it in the latter.

651. Let P (*fig.* 293) now be any point at the circumference of the equator of the external spheroid $adbe$, and the gravity at P towards the internal spheroid ADBE will be to the gravity at P towards the external solid as CA $\times$ CD^2 to Ca $\times$ Cd^2, or as the quantities of matter in these spheroids, the generating ellipses being supposed to have the same centre and *focus*, as in art. 649. To demonstrate this, PC being supposed perpendicular to the meridian plane $adbe$, let it meet DpE the circumference of the equator of the internal solid in p; and let the sections PZC, pVC perpendicular to the plane $adbe$ intersect it in CZ and CV, so that Zr and VR perpendicular to Cd may be always in the same ratio to one another as Ca to CA, and the elliptic sections PCZ, pCV will have the same excentricity, or CD^2 — CV^2 will be equal to Cd^2 — CZ^2. For let rZ and RV produced meet the circles dgh and DGH in g and G respectively, and Cd^2 — CZ^2 will be equal to gZ $\times$ $\overline{Zr + gr}$, which is to ha $\times$ $\overline{aC + hC}$ or Cd^2 — Ca^2 as Zr^2 to Ca^2. In the same manner CD^2 — CV^2 is to CD^2 — CA^2 as VR^2 is to CA^2. But Cd^2 — Ca^2 is equal to CD^2 — CA^2, and Zr^2 is to Ca^2 as VR^2 to CA^2, by the supposition; consequently Cd^2 — CZ^2 is equal to CD^2 — CV^2. Let the sections PCZ, pCV move into the places PCz, pCv, and it fol-

lows from what was shown in the last article, that the gravity at P towards the slice contained by the planes PZC, PzC is ultimately to the gravity at P towards the slice contained by pVC and pvC in the compound ratio of $CZ^2 \times CP$ to $CV^2 + Cp$ and of the angle ZCz to Vcv, or (because the areas of sectors are in the compound ratio of the squares of the rays and of the angles of the sectors) as $CZz \times CP$ to $CVv \times Cp$. But because Zr is always to VR as Ca to CA, and consequently Cr is to CR as Cd to CD; the area CaZr is to CAVR, and CaZ to CAV, as $Ca \times Cd$ to $CA \times CD$, and CZz to CVv in the same ratio. Therefore the gravity at P towards the slice terminated by the planes PZC and PzC is always to the gravity at P towards the slice terminated by pVC and pvC in the invariable ratio of $Ca \times Cd^2$ to $CA \times CD^2$; and the gravity at P towards the whole external solid is to the gravity at P towards the whole internal solid in the same ratio, or as the content of the former to the content of the latter solid.

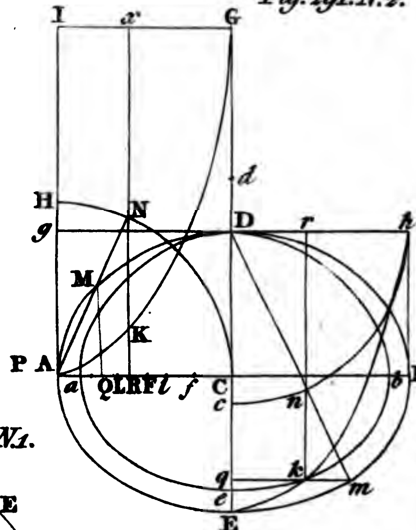
652. Hence to measure the gravity towards any oblate spheroid ADBE of an uniform density, at any distance CP in the plane of its equator produced, let F be the focus of a section of the solid through its axis, describe from the centre F with a radius equal to the distance CP an arch intersecting the axis in a, and from a as centre describe with the same radius the arch FO meeting CB in O; then the gravity at P towards the spheroid ADBE will be measured by $\frac{2CA \times CD^2}{CF^3} \times \frac{FCO}{CP}$; because this gravity is to the gravity at P towards the external solid $adbe$ (which is measured by $\frac{2Ca \times Cd^2}{CF^3} \times \frac{FCO}{CP}$, by article 646), as $CA \times CD^2$ to $Ca \times Cd^2$, by the last article. The gravity at P towards ADBE is to the gravity at a towards it as FCO to $CP + \overline{CF - CS}$, by art. 650. And if the density of the spheroid $adbe$ be supposed to vary from the surface to the centre, but so as to be always the same in the different parts of the same surface generated by any ellipse ADB that has always the same centre and focus with adb, the gravity at the equator of the solid $adbe$ will be to the gravity at the pole a in the same ratio as if the density of this spheroid was uniform.

653. The

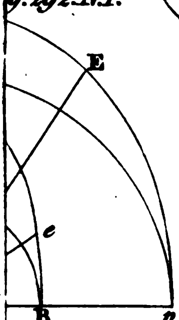
Plate XXXIII. Pa.130. Vol.II.

...1.N.1.

Fig.291.N.2.



...g.292.N.1.



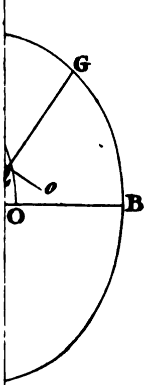
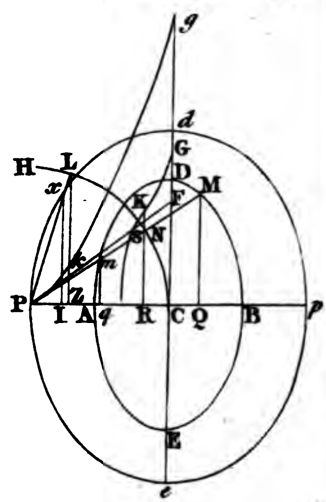


Fig.292.N.2.Art.649.



653. The rest remaining as in art. 651, suppose the solid not to be a spheroid, or Cp to be greater or less than CD, but so that the difference of the squares of Cp and CD be equal to the difference of the squares of CP and Cd, that the sections DpC, dPC may be still ellipses that have the same centre and focus; and if we suppose the sections PCZ, pCV to be always ellipses that have PC and CZ, pC, and CV for their respective axes, the distances of their *foci* from the centre C will be always equal, as before; and it will appear in the same manner, that the gravity at P towards the external solid will be to the gravity at P towards the internal solid as $Ca \times Cd \times CP$ to $CA \times CD \times Cp$.

654. Let x be any point in the surface of the spheroid *adbe*, which is supposed to be generated by an ellipse that has the same centre and focus with ADBE, as formerly, and the gravity at x towards the internal solid ADBE will be to the gravity at x towards the external spheroid *adbe* either accurately or nearly when the spheroids differ little from spheres, as $CA \times CD^2$ to $Ca \times Cd^2$, or as the content of the external to the content of the internal solid. When x is at the pole of the spheroid, or at the circumference of the equator, this appears from art. 650 and 652, and in other cases it may be deduced from the last article; but we proceed to the application of those theorems to enquiries that relate to the planetary system.

655. The gravity towards the spheroid ADBE (*fig.* 294) at the pole A being represented by A, at the equator by D, and the centrifugal force at D by V, as formerly; if the density of the spheroid be uniform, D will be to A as the area of the segment FCO to $CD \times \overline{CF - CS}$, by art. 646, that is (by the series usually given for the mensuration of circular segments and arches, the proof of which we are to give in the second book); b, a, and c, being supposed to represent CD, CA, and CF respectively as $1 + \frac{3c^2}{10b^2} + \frac{9c^4}{56b^4}$, &c. to $1 + \frac{2c^2}{5b^2} + \frac{8c^4}{35b^4}$, &c. Therefore $Db - Aa$, or Vb, will be to Db, or V will be to D, as $\frac{2c^2}{5b^2} + \frac{9c^4}{35b^4}$, &c. to $1 + \frac{3c^2}{10b^2} + \frac{9c^4}{56b^4}$, &c. And hence when the ratio of c to b is given, the ratio of V to D may be determined

to any degree of exactness, at pleasure. When the ratio of V to D is given, and thence the ratio of c to b is required, let V be to D as 1 to m, and c^2 to b^2 as z to 1, then $\frac{2z}{5}+\frac{9zz}{35}$, &c. will be to $1+\frac{3z}{10}+\frac{9zz}{56}$, &c. as 1 to m; from which it follows (by the methods for inversion of series) that z is equal to $\frac{5}{2m}-\frac{15}{7mm}$, &c. This series may be continued at pleasure; but when the spheroid differs little from a sphere, z will be nearly equal to $\frac{5}{2m+1\frac{5}{7}}$, and c^2 to b^2 nearly as 5V to $2D+\frac{12V}{7}$, consequently, in this case, the excess of the semidiameter of the equator above the semiaxis is to the mean semidiameter nearly as 5V to $4D-\frac{11V}{7}$.

656. The ratio of z to 1, or cc to bb, may be discovered several ways, without having observations made at the equator of the spheroid. For this end the two following properties of the ellipse are subjoined. Let PK perpendicular to the ellipse at any point P meet CD in K. Let the sine and co-sine of the angle PKD (which is the latitude of the place P) be denoted by S and K respectively, the radius being unit. Then PK^2 will be to CA^2 as CA^2 to $CA^2+CF^2\times KK$, or as CA^2 to $CD^2-CF^2\times SS$. For Pd being perpendicular to CD in d, dK will be to dC as CA^2 to CD^2, by art. 627, and dC^2 being to CA^2-Pd^2 as CD^2 to CA^2, dK^2 is to CA^2-Pd^2 as CA^2 to CD^2; consequently CA^2-PK^2 is to dK^2 as CF^2 to CA^2; and since dK^2 is to PK^2 as KK to 1, CA^2-PK^2 is to PK^2 as $CF^2\times KK$ to CA^2, and PK^2 to CA^2 as CA^2 to $CA+CF^2\times KK$, or as CA^2 to $CD^2-CF^2\times SS$.

657. The ray of curvature at any point P is always in the triplicate ratio of the perpendicular PK. For let CG be the semidiameter conjugate to CP, and because the ray of curvature at P is as the cube of CG, by art. 374, and PK is inversely as PZ the perpendicular to CG in Z (art. 627) which is inversely as CG, it follows that PK is as CG, and that the ray of curvature

ture at P is as the cube of PK. Hence the ray of curvature, or a degree upon the meridian, at any latitude P, is in the triplicate ratio of PK, or of the force with which a body descends towards the spheroid at P, by art. 637.

658. The magnitude of the earth is usually determined from that of a degree upon the meridian. This however gives us only the ray of curvature at that place of the meridian, or the radius of a sphere that would have all its degrees equal to that degree; and the centrifugal force derived from thence, and from the period of the earth's revolution upon its axis, is the centrifugal force at the equator of such a sphere when it is supposed to revolve on its axis in the same time with the earth. In order to derive the ratio of cc to bb in the spheroid from the observations made in any latitude P, let g represent the force with which a body is found by observation to descend towards the earth at P, v the centrifugal force at the equator of a sphere that has its degrees equal to the degree which we suppose to be measured at P, and that revolves on its axis in the same time with the spheroid; and, the radius being supposed equal to unit, let the sine of the latitude of P be S. Then CF^2 will be to CD^2, or cc to bb, nearly as $5v$ to $2g - \frac{9v}{7} + 5SSv$. For let Po the ray of curvature at P meet CD in K, PK be represented by L, and the ratio of g to v by that of n to unit. Then (by the last article) Po is to the ray of curvature at A, or $\frac{bb}{a}$, as L^3 to a^3; V is to v as DC to Po, the times in which the spheroid ADBE and the sphere of the radius Po are supposed to revolve being equal; consequently V is to v as a^4 to bL^3. But g is to A the gravity at the pole as L to a, by art. 637, and A to D $-$V as b to a, by art. 641, consequently g is to D$-$V as Lb to aa; and m, or $\frac{D}{V}$, equal to $\frac{aag}{LbV} + 1$, or $\frac{LLg}{aav} + 1$, or $\frac{nLL}{aa} + 1$, that is (art. 656), $\frac{naa}{bb - ccSS} + 1$. Therefore $n + 1 - nz - SSz$ is to $1 - SSz$ as m to unit, or (art. 655) as $1 + \frac{3z}{10} + \frac{9zz}{56}$, &c. to $\frac{2z}{5} + \frac{9zz}{35}$, &c. From which it follows,

 that

that when n is a large number, z is nearly to unit, or cc to bb, as 1 to $\frac{2n}{5} + \frac{1}{10} + SS + \frac{9nz}{35} + \frac{2nz}{5}$, or (because z is nearly equal to $\frac{5}{2n}$) as 1 to $\frac{2n}{5} + SS - \frac{9}{35}$, and therefore ($n$ being to 1 as g to v) as $5v$ to $2g + 5SSv - \frac{9v}{7}$. Hence the ratio of cc to bb is determined from the magnitude of a degree measured on the meridian in any latitude, and the length of the *pendulum* that vibrates in a given time in the same latitude (the earth being supposed of an uniform density), by computing v from the former, and g from the latter. At the equator this ratio is that of $5v$ to $2g - \frac{9v}{7}$, at the poles, that of $5v$ to $2g + \frac{26v}{7}$.

659. The ratio of c^2 to b^2 (*fig.* 293), may be likewise discovered from what has been demonstrated, by comparing the gravity of a satellite that revolves about the spheroid in the plane of its equator with the centrifugal force at D. Let Cd be any distance in the plane of the equator, and let Ca be taken upon the axis so that aF may be equal to Cd; from the centres a and A describe the arches FO and Fo meeting CB in O and o, and the gravity at d will be to the gravity at D as FCO × CD to FCo × Cd (by art. 646), or, supposing Cd to be represented by d, in the compound ratio of b^2 to d^2, and of $1 + \frac{3c^2}{10d^2} + \frac{9c^4}{56d^4}$, &c. to $1 + \frac{3c^2}{10b^2} + \frac{9c^4}{56b^4}$, &c. It appears from this, that the gravity towards an oblate spheroid decreases in the plane of its equator in a greater ratio than the square of the distance from the centre of the spheroid increases. Hence the periodic times of the satellites of *Jupiter* ought to increase in a greater proportion than according to *Kepler*'s law, or the sesquiplicate ratio of their distances from the centre of *Jupiter*; but the variation from his law will hardly be sensible even in the nearest satellites. In like manner, the gravity towards an oblate spheroid decreases in the axis produced in a less ratio than that in which the square of the distance

distance from the centre increases. For, the right lines aF and AF being supposed to meet the arches Cf and CS described from the centres a and A in f and S, the gravity at a towards the spheroid ADBE will be to the gravity at A towards the same spheroid as $CF - Cf$ to $CF - CS$, that is, Ca and CA being represented by e and a, in the compound ratio of a^2 to e^2, and of $1 - \frac{3c^2}{5e^2} +$, &c. to $1 - \frac{3c^2}{5a^2}$, &c. It appears in the same manner, that the gravity towards an oblong spheroid decreases in the plane of the equator in a less ratio than that in which the squares of the distances from the centre increase, but in a greater ratio in the axis produced from the pole.

660. Let N be to 1 in the compound ratio of the cube of Cd to the cube of CD, and of the square of the time in which the spheroid revolves on its axis to the square of the time in which a satellite revolves about the spheroid in the plane of its equator at the distance Cd; and let Cd be to CD as M to unit; then cc will be to bb nearly as 5 to $2N + \frac{45}{14} - \frac{3}{2MM}$. For, by the last article, the gravity at d is to the gravity at D in the compound ratio of 1 to MM and of $1 + \frac{3z}{10MM}$, &c. to $1 + \frac{3z}{10}$, &c. But the gravity at D is to V as $1 + \frac{3z}{10}$, &c. to $\frac{2z}{5} + \frac{9zz}{35}$, &c. consequently the gravity at d is to V, or $\frac{N}{MM}$ is to unit, in the compound ratio of 1 to MM and of $1 + \frac{3z}{10MM}$, &c. to $\frac{2z}{5} + \frac{9zz}{35}$, &c. Therefore $1 + \frac{3z}{10MM}$, &c. is equal to $\frac{2Nz}{5} + \frac{9Nzz}{35}$, &c. From which it follows, that when we may neglect the terms of the equation that involve the higher powers of z, it is equal to $\frac{5}{2N} - \frac{225}{56NN} + \frac{15}{8MMNN}$, &c. or z is nearly to unit, or cc to bb, as 1 to $\frac{2N}{5} + \frac{9}{14} - \frac{3}{10MM}$, and the excess of the semidiameter of the equator

above the semiaxis is to the mean semidiameter as 5 to $4N + \frac{10}{7} - \frac{3}{MM}$ nearly.

661. To apply those theorems to the earth, a degree of the meridian about the latitude of *Paris* is 57060 *toises* according to Mr. *Picart*; consequently if the earth was a perfect sphere, its radius would be 1961,5783 *French* feet, and a body at the equator of such a sphere would describe 1430 . 4 feet in a second of time by the diurnal motion, the versed sine of which is 7.510148 lines. Then because a *pendulum* that vibrates in a second at *Paris* (by the observations made lately by Mr. *De Mairan*) is 440.57 lines; and the space described by a body that descends freely by its gravity in any time is to the length of a *pendulum* that vibrates in the same time in the duplicate ratio of the semicircle to its diameter, by art. 405, it follows that in that latitude a body would describe by its gravity about 2172 . 9 in a second of time, and that v is there to g (according to the notation in art. 658), as 7 . 510148 to 2172 . 9 . or as 1 to 289 . 3. From which it follows by art. 658 that cc is to bb as 1 to 116, and that the excess of the semidiameter of the equator above the semiaxis is about $\frac{1}{231}$ of the mean semidiameter. If the degree of the meridian near to *Paris* be greater than 57060 *toises*, the ratio of this excess to the mean semidiameter will be greater almost in the same ratio; but though this degree be 57183 *toises* (as it is said to be found by some late observations), that excess will not be above $\frac{1}{230}$ of the mean semidiameter. By the mensuration and observations of the members of the Royal Academy of Sciences at *Paris* made near the polar circle, v is to g there as 1 to 287 . 8 . as will appear by comparing in the same manner the degree measured by them with the length of the *pendulum*, which, by their observations, vibrates at *Pello* in a second of time. From which cc is to bb as 1 to 115 . 9 . and almost the same excess of the semidiameter of the equator above the semiaxis arises as from the observations at *Paris*. This ratio may be likewise determined from the distance and periodic time of the moon, compared with the time in which the earth revolves on its axis, and thence finding the

ratio

ratio of N to unit, according to art. 660. By this computation the difference of the semidiameters of the earth is nearly the same as by the former. And these agree nearly with Sir *Isacc Newton*'s, according to which the semidiameter of the equator is to the semiaxis as 230 to 229.

662. But supposing the earth to be a spheroid, according to what was demonstrated above, upon the supposition that the density is uniform from the surface to the centre, if we compute the difference of those semidiameters of the earth from the lengths of *pendulums* that have been found to vibrate in equal times in different latitudes; or from the increase of the degree of the meridian from *Paris* to the polar circle, as it has been determined lately; the difference of these semidiameters will be found to be considerably greater than $\frac{1}{231}$ of the mean semidiameter. Let L and l denote the lengths of two such *pendulums* at two places P and p, and, the radius being unit, let S and $ſ$ represent the respective sines of the latitudes of P and p (that is, of the angles PKD, pkD), then cc will be to bb as $LL - ll$ to $LLSS - llſſ$. For PK^2 is to pk^2 as $1 - \frac{ccſſ}{b^2}$ to $1 - \frac{ccSS}{bb}$, by art. 656. The space described in a given time by a body descending freely is as the gravity; and it follows by art. 408, that the length of a *pendulum* that vibrates in a given time is likewise as the gravity; consequently LL is to ll as PK^2 to pk^2 by art. 647, or as $1 - \frac{ccſſ}{bb}$ to $1 - \frac{ccSS}{bb}$. Therefore cc is to bb as $LL - ll$ to $LLSS - llſſ$. Hence if L be to l as $1 + u$ to $1 - u$, cc will be to bb nearly as $4u$ to $SS - ſſ + 2uSS + 2uſſ$.

663. If a degree upon the meridian at P be to a degree at p as G to g, cc will be to bb as $G^{\frac{2}{3}} - g^{\frac{2}{3}}$ to $G^{\frac{2}{3}}SS - g^{\frac{2}{3}}ſſ$; because G is to g as the ray of curvature at P to the ray of curvature at p, that is, as PK^3 to pk^3, by art. 657; consequently $G^{\frac{2}{3}}$ is to $g^{\frac{2}{3}}$ as $1 - \frac{ccſſ}{bb}$ to $1 - \frac{ccSS}{bb}$, and cc to bb as $G^{\frac{2}{3}} - g^{\frac{2}{3}}$ to $G^{\frac{2}{3}}SS - g^{\frac{2}{3}}ſſ$. This rule for finding the ratio of

of cc to bb (whence the ratio of $bb - cc$, or aa, to bb is easily computed) is accurate, and is founded on no particular theory of gravity, but on the supposition that the earth is a spheroid only. When the spheroid differs little from a sphere, let the degree at P be to the degree at p as $1 + x$ to $1 - x$, and cc will be to bb nearly as $\frac{4x}{3}$ to $SS - ff + \frac{2SSx}{3} + \frac{2ffx}{3}$.

664. For example, the length of the *pendulum* that vibrates in a second of time at *Pello* latit. 66°. 48′. is 441. 17 by the observation * of Mr. *De Maupertuis*, &c. The *pendulum* that vibrates in the same time at *Paris, Latit.* 48°. 50′. 10″. is 440. 57 lines. Suppose therefore L to be to l as 441. 17 to 440. 57, or as $1 + \frac{30}{44087}$ to $1 - \frac{30}{44087}$, and by computing from either of the rules in art. 662, cc will be to bb as 1 to 102. 8. By comparing in the same manner the observations made in *Jamaica* by *Colin Campbell*, Esq. †, and at *London* by Mr. *Graham*, cc is to bb nearly as 1 to 95; and by computing from some other observations of this kind, ‡ this ratio is found still greater; which ought to be that of 1 to 116, if the earth was of an uniform density, by art. 660.

665. The degree that cuts the polar circle was found to be 57438 *toises*, and the middle of the arch was in latit. 66°. 19′. 34″. The degree measured by Mr. *Picart*, allowing the correction made lately by Mr. *De Maupertuis*, is of 57183 *toises*, and the middle of the arch that was measured is in latit. 49°. 21′. 24″. Suppose therefore G to g as 57438 to 57183, or as $1 + \frac{1275}{573105}$ to $1 - \frac{1275}{573105}$, and by the rules in art. 663, cc will be to bb as 1 to 89 $\frac{1}{5}$. Hence, b is to a or the semidiameter of the equator to the semiaxis, in the subduplicate ratio of 89 $\frac{1}{5}$ to 88 $\frac{1}{5}$, or nearly as 178 $\frac{1}{5}$ to 177 $\frac{1}{5}$, and consequently the difference of those semidiameters is about 22 miles, which if the density of the earth was uniform ought to be 17 miles only. If the correction of Mr. *Picart's* arch be not allowed, the difference of those semidiameters will be considerably greater.

* Figure of the earth, Book 3, Ch. 6. § 6. † Phil. Trans. N. 432. ‡ Mem. de l'Acad. 1735.

666. From

666. From these observations, there is ground to think that the variation of the density of the internal parts of the earth is considerable; and to enable us to form some judgment of this, it may be of use to enquire what proportions of the semidameters DC and AC, and of the gravitation at A to the gravitation at D, arise, when the density is supposed to increase or decrease towards the centre; or even when the earth is supposed to be hollow with a *nucleus* included, according to the ingenious hypothesis advanced long ago by Dr. *Halley*. If the density was uniform, the increase of *gravitation* (by which we shall understand with Mr. *De Maupertuis* in what follows the force with which a body actually tends downwards, or the excess of the gravity above the centrifugal force) from the equator to the poles ought to be in the same proportion to the mean gravitation as the difference of the semidiameters DC and AC to the mean semidiameter; because A is to D—V as DC to AC, by art. 641, and the excess of A above D — V to half their sum, as DC — AC to $\frac{1}{2}$ DC + $\frac{1}{2}$ AC. If we suppose new matter to be added at the centre, or the density to be increased there, the attraction of this new matter will add more to the gravity at the pole than at the equator, the distance being less, and may account for a greater increase of gravitation from D to A than arises from the hypothesis of an uniform density, as Sir *Isaac Newton* has justly observed. But this will not account for a greater difference of the semidiameters DC and AC. Supposing the columns to be fluid (after Sir *Isaac's* manner), and to have sustained each other before the new matter was added at the centre, the attraction of this new matter will add more to the gravity of the longer column DC than of CA; and though we suppose the centrifugal force at D to be increased till it be in the same ratio to the whole gravity at D as before, the column CD will be more than a counterpoise for CA, till CD and CA come nearer to an equality, and the figure nearer to a sphere. For let d represent the increment of the gravity at D from the attraction of that new matter, N the increment of the gravitation of the column AC arising from the same attraction, and the increment of the centrifugal force at D being represented by v, let v be to d as V to D, that the ratio

tio of 1 to m (or Vv to $D + d$) may remain the same as before; then $\frac{Aa}{2} + N$ will represent the whole gravitation of the column AC, and $\overline{D-V} \times \frac{b}{2} + N + d \times \overline{b-a} - \frac{vb}{2}$ the gravitation of DC; but $\frac{1}{2}$ Aa is supposed equal to $\frac{1}{2}$ $b \times \overline{D-V}$; and $\frac{vb}{2}$ being equal to $\frac{dVb}{2D}$, or (because c^2 is to b^2 as 5V to 2D, nearly, by art. 655) $\frac{dc^2}{5b}$, which stands less than $d \times \overline{b-a}$, or $\frac{dc^2}{b+a}$, in the ratio of 2 to 5, nearly, it follows that the gravitation of DC is now greater than that of AC; so that these columns cannot balance each other, unless the fluid subside at D and rise at A. If the new matter be in the form of a sphere about the centre C, it is shown in the same manner, that the column AC will not balance DC; and the same will appear afterwards, when the new matter is supposed to be formed into a spheroid similar and concentric to ADBE.

667. On the other hand, if we suppose the density to be less at the centre, or some matter to be taken away there, the column DC will no longer balance or sustain the column AC; and the fluid in the canal ACD will not be in *æquilibrio* till it rise at D and subside at A; that is, till the figure vary more from a sphere than in the case when the density was supposed uniform: for supposing the decrement of the gravity at D in consequence of the rarefaction of the matter at the centre to be represented by d, and the decrement of the gravity of the whole column AC by N; let v the decrement of the centrifugal force be such, that $V-v$ may be now to $D-d$ in the same ratio as V was to D; then $\frac{Aa}{2}$ —N will represent the gravitation of the column AC, and $\frac{\overline{D-V+v}}{2} \times b - N - d \times \overline{b-a}$ the gravitation of CD. But $\frac{vb}{2}$ being less than $d \times \overline{b-a}$, as in the last article; and $\frac{Aa}{2}$ equal to $\frac{\overline{D-V}}{2} \times b$, because the columns were supposed to sustain each other before the matter at the centre was

was taken away; it appears that the column AC is now more than a counterpoise for DC. Thus the rarefaction of the matter at the centre will account for a greater difference of the semidiameters DC and AC, or a greater variation from the spherical figure, than the hypothesis of an uniform density. But it will not account for a greater increase of gravitation from the equator to the poles. On the contrary the increase of gravitation will be less in this case than when we suppose the density uniform. For since $A - D + V$ is to $A + D - V$ as $b - a$ to $b + a$, that is, as 5 to $8m$, nearly, by art. 655, the increase of gravitation from the equator to the poles is nearly to the mean gravity (which we shall call G) as 5 to $4m$, when the density of the spheroid is uniform. But when the matter about the centre is supposed to be rarefied, as above, let d be to G as 1 to r; and the gravity at A being $A - \frac{db^2}{a^2}$, and the gravitation at D equal to $D - d - V + v$, the difference of which is to half their sum as $A - D - \frac{dc^2}{a^2} + V - v$ to $\frac{1}{2} A + \frac{1}{2} D - \frac{db^2 + da^2}{2a^2} - \frac{1}{2} V + \frac{1}{2} v$; it follows (because $A - D + v$ is to $2G$ as $b - a$ to $b + a$ or 5 to $8m$, c^2 to a^2 as 5 to $2m$, and v to G as 1 to rm nearly), that the increase of gravitation from the equator to the poles will be in this case to the mean gravitation nearly as $5r - 14$ to $4mr - 4m + 2$, or as $5 - \frac{9}{r-1}$ to $4m + \frac{2}{r-1}$, which is a less ratio than that of 5 to $4m$. And if we suppose the fluid to rise at D and subside at A, till the columns AC and DC sustain each other, the increase of gravitation from D to A will in this case be to the mean gravitation in a less ratio than before. The hypothesis therefore of a greater density towards the centre may account for a greater increase of gravitation from the equator to the poles than that of an uniform density, but not for a greater increase of the degrees of the meridian: and the hypothesis of a less density towards the centre may account for a greater increase of the degrees of the meridian, but not for a greater increase of the gravitation, supposing always (after Sir *Isaac Newton*'s manner) the columns DC and AC to extend from the surface to the centre, and there to sustain each other.

This

This is likewise the result of our computations (some of which we are to subjoin), when we have supposed the density to increase or decrease continually from the surface of the spheroid ADBE to the centre, so as to be uniform in the different parts of any one similar and concentric elliptic surface; and in several other cases. And hence there seems to be some foundation for proposing it, as a *Query*, Whether the internal constitution of the parts of the earth, above-mentioned, that was proposed by Dr. *Halley* for resolving some of the phænomena of the magnetick needle, will not be found to account in a probable manner for the increase of gravitation, and at the same time of the degrees of the meridian from the equator to the poles; as these have been determined by the best observations hitherto. The grounds upon which we mention this will appear better from what follows.

668. Let ADBE (*fig.* 295) be a section of a spheroid through its axis AB, F the *focus*, and FO an arch described from the centre A, as formerly, meeting CB in O; let *adbe* be any similar concentric ellipse, *f* its *focus*, *f*Z a parallel to the axis meeting the arch FO in Z, and ZV a perpendicular to the axis in V. Suppose the density to be always the same over the surface generated by any ellipsis *adb* about the axis AB, however variable it may be in different elliptic surfaces; and let *e* represent the density at the surface *adbe*. Then if VK be taken upon VZ in the same ratio to VZ as *e* is to CD, and the ordinate VK generate the area OKHC, the gravity at D towards the whole spheroid ADBE will be measured by $\frac{2CD^2 \times CA}{CF^3} \times OKHC$. For let *lmnr* be another similar and concentric ellipsis, *x* its *focus*, *xz* parallel to AB meet FO in *z*, *zv* be perpendicular to AB in *v*; then (by art. 652) the gravity at D towards the solid generated by the annular space bounded by *adb* and *lmn* revolving about the axis AB, of the density *e*, will be measured by $\frac{2CD^2 \times CA \times e}{CF^3} \times \frac{ZVvz}{CD}$; which, when *al* is continually diminished, is ultimately equal to $\frac{2CD^2 \times CA \times e}{CF^3} \times \frac{ZV \times Vv}{CD}$ or (by the supposition) to $\frac{2CD^2 \times CA}{CF^3} \times VK \times Vv$; consequently the

gravity

gravity at D towards the whole spheroid ADBE is measured by $\frac{2CD^2 \times CA}{CF^3} \times OKHC$; and is to the gravity at D towards the spheroid ADBE, when its density is supposed uniform, and represented by E, as OKHC × CD to FCO × E. For example, if the density in the ray CD at any point *d* be inversely as C*d* the distance from the centre, the gravity at D towards this spheroid will be to the gravity at D towards a spheroid of an uniform density equal to that of the former at D, as CF × CO to the area FCO; because if E represent the density at D, VK will be to E as VZ (or C*f*) to C*d*, or as CF to CD, and the area OKHC × CD equal to E × CF × CO. In this case the gravity is the same in all parts of the column DC. In the same manner, when the density at *d* is inversely as the square of the distance C*d*, the gravity at D towards such a spheroid is to the gravity at D towards the spheroid when its density is uniform and equal to that of the former at D, as $CF^2 \times CO$ to FCO × CD: and the gravity at any point *d* in the column CD is inversely as the distance C*d*.

669. In like manner, let *fk* perpendicular to CD at *f*, the *focus* of *adbe*, be to *e* as Cf^2 to Af^2, and the ordinate *fk* generate the area C*ko*F; and the gravity at A towards the spheroid ADBE will be measured by $\frac{2CD^2 \times CA}{CF^3} \times CkoF$. This is demonstrated in the same manner from art 650. The gravity towards such a spheroid at any point in its axis, or in the plane of its equator produced without the solid, may be determined in the same manner.

670. Suppose, for example, that the density in any semidiameter is as the distance from C, and the density at the surface being represented by E, *e* will be to E as C*d* to CD, and VK to VZ as E × C*d* to CD^2, or (because C*d* is to CD as ZV to CF) as E × ZV to CD × CF; and VK will be to E as ZV^2, or $AO^2 - AV^2$, to CD × CF; consequently if AM perpendicular to AO be to AO, or CD, as E is to CF; and a parabola be described upon the axis MA that shall have its vertex in M and pass through O, OKH will be a portion of this parabola; and the area OKHC will be found equal to

E ×

$E \times \frac{3CD^2 \times CO - CD^3 + CA^3}{3CD \times CF}$, or (according to the notation in art. 655), to $\frac{Ec^3}{3b} \times \frac{2b+a}{b+a^2}$. Therefore the gravity at D towards such a spheroid will be measured by $\frac{2baE}{3} \times \frac{2b+a}{b+a^2}$. The gravity at d is to the gravity at D in the compound ratio of Cd to CD and of the density at d to the density at D, and consequently as Cd^2 to CD^2. Therefore if the gravity at D be represented by Q, and Cd by z, the gravity at d will be represented by $\frac{Qz^2}{b^2}$, the density at d by $\frac{Ez}{b}$, and the gravity of the column DC will be measured by an area upon the base CD that has its ordinate at any point d equal to $\frac{QEz^3}{b^3}$; and this area is equal to $\frac{1}{4}$QEb. Any distance in the plane of the equator, as Cp, greater than CD being represented by d, and $\sqrt{d^2 - c^2}$ by a, the gravity at p will be measured by $\frac{2b^2 aE}{3d} \times \frac{2d+a}{d+a^2}$; as will appear in the same manner.

671. In the same spheroid, fk is to be taken to E as Cf^3 to A$f^2 \times$ CF; and if L denote the logarithm of the ratio of DC to AC, the *modulus* being AC, the area CkoF will be found equal to $\frac{E}{2CF} \times \overline{CF^2 - 2AC \times L}$; and the gravity at A towards the spheroid will be measured by $\frac{b^2aE}{c^4} \times \overline{c^2 - 2aL}$. The gravity at a towards it will be to the gravity at A in the compound ratio of the density at a to the density at A and of Ca to CA, that is, as Ca^2 to CA^2. Therefore if q denote the gravity at A, and Ca be represented by u, the gravity at a will be $\frac{qu^2}{a^2}$, and the density at a will be $\frac{Eu}{a}$; consequently the gravity of the column AC will be measured by $\frac{1}{4}$ qEa.

672. Let V represent the centrifugal force at D, arising from the rotation of the spheroid on its axis, the centrifugal force

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at d will be $\frac{Vz}{b}$, and, the density at d being $\frac{Ez}{b}$, the quantity to be subducted from the gravity of the column DC, on this account, will be measured by an area on the base CD that has the ordinate at any point d equal to $\frac{VEzz}{bb}$; and this area being equal to $\frac{1}{3}$ VEb, the gravitation of the column DC is $\frac{1}{4}$ EbQ — $\frac{1}{3}$ EbV, or (supposing V to be to Q the gravity at D as 1 to m, as formerly) EbQ $\times \frac{3m-4}{12m}$.

673. If we now suppose (after Sir *Isaac Newton's* manner) the columns DC and AC to be fluid, and to sustain each other at C, we shall have bQ $\times \frac{3m-4}{12m}$ equal to $\frac{aq}{4}$, or b to a as q to Q $\times \frac{3m-4}{3m}$. But when the spheroid differs little from a sphere, Q and q may be considered as equal; for by art. 670 and 671 (supposing CD to be equal to $1+x$, and CA to $1-x$), Q will be to q as $\frac{2}{3} \times \frac{2b+a}{b+x^{2}}$ or $\frac{1}{1} + \frac{x}{3}$, to $\frac{b}{x^{3}} \times \overline{cc-2aL}$; which last being likewise expressed by x, those terms only will be found different that involve the second and higher powers of x. Therefore b is to a nearly as $3m$ to $3m-4$, and $b-a$ to b nearly as 4 to $3m$, that is, as 4V to 3Q. And in this case the excess of the semidiameter of the equator above the semiaxis is greater than when the density is supposed uniform in the ratio of 16 to 15, the ratio of V to Q being supposed the same as that of V to D was before. Let Q be to V as 289 to 1, as in the earth, and CD — CA will be to CD as 4 to 3 $\times$ 289, or as 1 to 216 $\frac{3}{4}$; consequently CD will be to CA as 216 $\frac{3}{4}$ to 215 $\frac{3}{4}$. This hypothesis, therefore, of a density that decreases as the distance from the centre decreases, might account for a greater excess of the semidiameter of the equator above the semiaxis, than that which results from the supposition of an uniform density; but it would not account for a greater increase of gravitation from the equator to the poles. For since the values of Q and q almost coincide in this case, it follows that the gravitation at the equator is to the gravity at

the pole as Q — V to Q, or as $m - 1$ to m, that is, as 288 to 289; whereas in the hypothesis of an uniform density, this ratio was that of 230 to 231. In like manner, by supposing the density to decrease in the same proportion as the cube of the distance from C, the ratio of DC to AC will be found to be that of 226 to 225, nearly, but the increase of gravitation will be less than in the former hypothesis.

674. It will be easy, from what has been shown, to measure the gravity at D and A towards a spheroid, when the depths from the surface being supposed to increase uniformly, the density increases likewise uniformly, till at the centre it become any multiple of what it is at the surface; and to determine the form of the ellipsis ADBE. Let L be taken upon CD produced upwards, so as that CL may be to LD as any number n to 1; and suppose the density at any point d to be always as Ld. Let e denote the density at the surface, and ne will represent the density at the centre. In this case, we may conceive the density of the spheroid at any orb *adbe*, as the difference of the densities of a spheroid of an uniform density ne, and of another spheroid that has the density at its surface equal to $\overline{n-1} \times e$, and its density decreasing downwards in the same proportion as the distance from C, as in the preceding articles; because the difference of those densities at any point d will be equal to $ne - \overline{n-1} \times \frac{Cd}{CD} \times e$ or $\frac{LC - Cd}{LD} \times e$, or $\frac{Ld}{LD} \times e$, which represents the density at d of the spheroid ADBE that we are now considering; consequently the gravity at any point towards this spheroid ADBE is equal to the difference of the gravities towards those two spheroids at the same point. Therefore if P denote the gravity at D towards a spheroid of an uniform density represented by ne, and Q denote the gravity at D towards the spheroid, whose density at D is $\overline{n-1} \times e$, and at any other point d is as Cd; then P — Q shall denote the gravity at D towards the spheroid ADBE, and (CD and Cd being represented by b and z as formerly) $\frac{Pz}{b} - \frac{Qz^2}{b^2}$ the gravity at d towards it. The density at d is represented by $ne - \overline{n-1} \times \frac{ze}{b}$;

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consequently the gravity of the column CD will be measured by an area upon the base CD, of an ordinate at d equal to $\frac{Pz}{b} - \frac{Qz^2}{b^2} \times \overline{ne - n - 1} \times \frac{ez}{b}$, that is, by $\frac{eb}{6} \times \overline{n+2} \times P - \frac{\overline{n+3}}{2} \times Q$. The value of P is $\frac{2bba}{c^3} \times ne \times \frac{FCO}{b}$, by art. 646; and the value of Q is $2bae \times \frac{n-1}{3} \times \frac{2b+a}{b+a^2}$, by art. 670. If it is required to determine the gravity towards this spheroid at any point p in the plane of its equator produced, describe from the centre F, with a radius equal to Cp, an arch intersecting the axis in p, and the arch Fo with the same radius from the centre p intersecting the axis in o; and the gravity at p towards the spheroid ADBE will be measured by $\frac{2CD^2 \times CA}{CF^3} \times \frac{FCo}{Cp} \times ne - \frac{2DC^2 \times CA}{3Cp} \times \frac{2Cp+Cp}{\overline{Cp+Cp}^2} \times \overline{n-1} \times e$.

675. The centrifugal force at D being represented by V, the centrifugal force at d will be $\frac{Vz}{b}$, and the density at d being $ne - \overline{n-1} \times \frac{ez}{b}$, the quantity to be subducted from the gravitation of the column DC, on account of the centrifugal force, will be measured by an area upon the base DC the ordinate at d being always equal to $\frac{neVz}{b} - \overline{n-1} \times \frac{eVzz}{bb}$; and this area is $ebV \times \frac{n+2}{6}$ (or supposing the ratio of the centrifugal force V to P — Q, the gravity at D to be represented by that of 1 to m) $eb \times \frac{P-Q}{m} \times \frac{n+2}{6}$. Therefore the gravitation of the column DC, by subducting this quantity, is reduced to $ebP \times \frac{n+2}{6} \times \frac{m-1}{m} - \frac{ebQ}{6} \times \overline{\frac{n+3}{2} - \frac{n+2}{m}}$.

676. Let p denote the gravity at the pole A towards a spheroid of an uniform density represented by ne, and q the gravity at A towards the other spheroid, the density of which in any column AC is as the distance from C; then $p - q$ will denote

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the gravity at A towards the spheroid ADBE (the density of which at any orb *adbe* is supposed to be as *Ld*), and by proceeding as in art. 674, the gravity of the column AC will be found to be measured by $\frac{ea}{5} \times \overline{\overline{n+2} \times p - \frac{n+3}{2} \times q}$. The value of p is $\frac{2b^2a}{c^3} \times \overline{CE - CS} \times ne$, and the value of q is $b^2ae \times \frac{n-1}{c^4} \times \overline{c^2 - 2aL}$, by art. 671.

677. The supposition of the *æquilibrium* of the columns DC and AC gives us $bP \times \frac{m-1}{m} - bQ \times \overline{\frac{n+3}{2n+4} - \frac{1}{m}}$ equal to $ap - aq \times \frac{n+3}{2n+4}$, or N being supposed equal to $\frac{n+3}{2n+4}$, $bP \times \frac{m-1}{m} - bQ \times \frac{Nm-1}{m}$ equal to $ap - Naq$; and b to a as $p - Nq$ to $\frac{m-1}{m} \times P - Q \times \frac{Nm-1}{m}$; so that $b - a$ will be to $b + a$, or (supposing b equal to $1 + x$, and a to $1 - x$) x to 1, as $p - P \times \frac{m-1}{m} - \frac{q}{m}$ to $p + P - 2NQ$, nearly; because Q and q may be considered as equal, by what was observed in art. 673, and m is supposed to be a large number. From this it will be found (by substituting for p, P, and Q their values, from art. 674 and 676, and neglecting the terms where the index of x is greater than unit, and where x is divided by m) that $\frac{16nx}{5} - N \times \overline{n-1} \times x$ is equal to $\frac{n+3}{6m}$, and (substituting for N its value $\frac{n+3}{2n+4}$) x equal to $\frac{5}{m} \times \frac{\overline{n+2} \times \overline{n+3}}{17nn+34n+45}$. The same value of x is found when n is a fraction; that is, l being taken upon CE, when the density at the centre is less than the density at D in the ratio of lC to lD, and the density at any point d is as ld. According as n is greater or less than unit, x is less or greater than $\frac{5}{8m}$; for x is equal to $\frac{5}{8m}$

$-\frac{45}{8m} \times \frac{\overline{3n+4} \times \overline{n-1}}{17nn+34n+44}$. Therefore the ratio of the centrifugal force at D to the gravity being given, the spheroid is found to differ less from a sphere, when the density increases towards the centre in the manner we have described above, than when the density is supposed uniform; but to vary more from a sphere when the density decreases towards the centre.

678. The increase of gravitation from the equator to the pole is to the mean gravitation as $p - \frac{m-1}{7m} \times P - \frac{Q}{m}$ to $\frac{4}{5}P + \frac{1}{5}p - Q + \frac{P-Q}{2m}$; that is, in the compound ratio of 1 to m, and of $25 \times \overline{n+1}^2 + 20$ to $17 \times \overline{n+1}^2 + 28$; or in the compound ratio of 5 to $4m$, and of $1 + 3 \times \frac{\overline{n+3} \times \overline{n-1}}{17nn+34n+28}$ to 1. Therefore the increase of gravitation from the equator to the poles is to the mean gravitation in a greater or less ratio than that of 5 to $4m$ (which is the ratio when the density is uniform) according as n is greater or less than unit; that is, according as the density increases or decreases towards the centre. And it appears from hence, and from the last article, that no supposition of this kind can account for a greater variation from the spherical figure, and at the same time for a greater increase of gravitation from the equator to the poles, than the hypothesis of an uniform density; if the columns AC and DC be supposed to extend from the surface to the centre, and be supposed to balance each other at C.

679. To mention some examples: if the density at the centre be double of what it is at the surface, or n be equal to 2, the excess of DC above AC will be to the mean semidiameter as 200 to 181 m; consequently in the earth (m being equal to 289) the semidiameter of the equator will be to the semiaxis as 262 to 261, and the gravitation at the equator to the gravitation at the poles as 213 to 214. If n be equal to 3, the difference of CD and CA will be to the mean semidiameter as 1 to m; and in the earth the semidiameter of the equator to the semiaxis as 289 to 288; in which case the gravitation at the equator will be to the gravitation at the poles as 206 to 207.

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If the density be as the distance below the surface, or the point L coincide with D, the difference of CD and AC will be to the mean semidiameter as 10 to 17 m; in the earth DC will be to AC as 492 to 491, and the gravitation at D to the gravitation at A as 196 to 197.

680. Suppose the density to be uniform from the surface ADBE to the similar concentric orb *adbe*, and to be uniform likewise from *adbe* to the centre; and the density within the orb *adbe* be to the density without it as $1 + e$ to 1. In this case the increase of gravitation from D to A will be greater than in the hypothesis of an uniform density; but supposing the columns AC and DC to sustain each other at C, and DC to be to dC as n to 1, then the excess of the semidiameter of the equator above the semiaxis will be to the mean semidiameter nearly in the compound ratio of 5 to $4m$, and of $n^5 + en^3 + en^2 + ee$ to $n^5 + en^3 + en^2 + een^2 + 3e \times \frac{nn-1}{nn+1}$; which compound ratio, when e is positive, is manifestly less than that of 5 to $4m$ (the ratio of the difference of CD and CA to the mean semidiameter when the density is supposed uniform), since n is necessarily greater than unit. This likewise holds, when there are three or more such orbs, providing the density be always greater within the orbs that are nearest to the centre.

681. Let us therefore now suppose the earth ADBE to be hollow with a *nucleus lmnr* included; let the convex and concave elliptic surfaces ADBE, *adbe* that bound the external part be similar; and first let *lmnr* be a sphere. Let CD be to Cd as n to 1, the area of the sphere *lmnr* to the area of the spheroid ADBE as 1 to N, the centrifugal force at D to the gravity as 1 to m; and the external part bounded by ADBE and *adbe* being supposed of an uniform density, if we suppose the columns Aa and Dd to gravitate equally, the excess of CD above CA will be to the mean semidiameter nearly in the compound ratio of $5n + 5$ to $2m$N, and of $n^3 + n^3\text{N} - \text{N}$ to $2n^4 + 2n^3 + 2n^2 - 3n - 3 - \frac{5n^3}{\text{N}}$; and the increase of gravitation from the equator to the poles will be to the mean

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gravitation nearly as $1 + \frac{n^5 - 30nn + 9 + 10n^2 N}{2n^4 + 2n^3 + 2n^2 - 3n - 3 - 5n^5 N} \times \frac{n+1}{2nn}$ to m. In this case the difference of the semidiameters CD and CA, and the increase of gravitation from D to A, may be both greater than when the density is supposed uniform, the ratio of 1 to m being supposed the same in both cases. For example, let n be supposed equal to 5, N to 45, and m to 289; then the semidiameter of the equator will be to the semiaxis as $180\frac{1}{2}$ to $179\frac{1}{2}$ nearly; and the increase of gravitation from the equator to the poles will be $\frac{1}{220}$ of the mean gravitation. If *lmnr* be a spheroid (as is more probable), and f the focus of a meridian section of it, let Cf be to the mean semidiameter of ADBE as 1 to r; and the rest remaining as in the former case, the difference of CD and CA will be nearly to the mean betwixt CD and CA as $5 \times \frac{n+1}{2m} \times \overline{n^3 - 1 + \frac{n^3}{N} - \frac{3n^5}{2rrN}} \times \overline{nn + n + 1}$ to $2n^4 + 2n^3 + 2n^2 - 3n - 3 - \frac{5n^5}{N}$. This ratio may be computed from the same principles, when the density is supposed to increase or decrease from ADBE to *adbe*. But, because the hypothesis of the equal gravitation of the columns A*a* and D*d*, as well as of an uniform density in the different parts of every elliptic orb similar and concentric to ADBE, may seem precarious, we shall not insist on the consequences that would follow from such a constitution of the internal parts of the earth, as we have here supposed. If we suppose the density to be uniform in the different parts of every orb *adbe* that is generated by an ellipse, which has always the same centre and *focus* with ADBE, but to vary in different orbs of this kind, the gravity at any point in CD or CA may be computed from the principles in art. 650 and 652. But the conclusions deduced from this hypothesis, when the density is supposed to increase towards the centre, agree no better with the phænomena than those in art. 677 and 678. By imagining the density to be greater in the axis than in the plane of the equator at equal distances from the centre, an hypothesis perhaps might be found that would account for most of the

phænomena; but as this may seem to be an improbable supposition, and it is not so easy to compute the consequences that would result from it, we shall insist on this subject no further. When more degrees shall be measured accurately on the meridian, and the increase of gravitation from the equator towards the poles determined by a series of many exact observations, the various *hypotheses*, that may be imagined concerning the internal constitution of the earth, may be examined with more certainty. We have always abstracted from any powers that may affect the gravitation, besides the mutual gravity of the particles and their centrifugal force.

682. The figure of the planet Jupiter is found to differ considerably from a sphere, by the observations of Astronomers, as well as by this theory. By Dr. *Pound's* observations, the distance of the fourth satellite is to the greatest semidiameter of Jupiter as 26,63 to 1, and its periodic time to the time in which Jupiter revolves on his axis as 24032,15 to 596. Therefore let N be to 1 in the compound ratio of the cube of 26,63 to 1, and of the square of 596 to 24032,15 according to art. 660, and N will be found equal to 11,615. By continuing the series in art. 660, one step further, the excess of the semidiameter of the equator above the semiaxis is to the mean semidiameter as 5 is to $4N + \frac{10}{7} - \frac{3}{MM} + \frac{4825}{336N}$, &c. M being equal to 26,63: consequently this ratio is that of 1 to 9,8; and the semidiameter of the equator to the semiaxis as 10,3 to 9,3, the density being supposed uniform; and this agrees with Sir *Isaac Newton's* computation. But the difference of those semidiameters, according to Mr. *Cassini*, is only $\frac{1}{15th}$ of Jupiter's semidiameter, and by Doctor *Pound's* observations is betwixt $\frac{1}{12th}$ and $\frac{1}{15th}$ of it. Hence, according to what was shown in art. 677, the density of Jupiter seems to increase towards the centre. We have abstracted from the effect of the gravitation of the fourth satellite towards the other satellites, and towards the atmosphere of Jupiter (if there is any); but the difference betwixt this computation and the observations cannot be imputed to these. It is nearly

ly the same ratio of the semidiameters of Jupiter that is found by computing from Dr. *Pound*'s observations of the elongation of the third satellite.

683. If we suppose the density of Jupiter to increase from the surface to the centre, in the manner described in art. 674, so as to become quadruple at the centre of what it is at the surface; then, by art. 677, CD being supposed equal to $1 + x$, and CA to $1 - x$, x will be nearly equal to $\frac{210}{453m}$. By computing from what was shown at the end of art. 674, m will be nearly to N as $n + 3 + \frac{14nx}{5} + 10x$ to $\overline{1+2x} \times \overline{n+3} + \frac{24nx}{5MM}$; and supposing n equal to 4, m will be equal to 12 nearly, x to $\frac{1}{25,8}$; and the semidiameter of the equator to the semiaxis as 13,4 to 12,4; which differs little from the mean ratio that results from Dr. *Pound*'s observations.

684. Sir *Isaac Newton* has found, that the mean density of Jupiter is to the mean density of the earth as 94 ½ to 400. If we suppose n equal to 4, as in the last article, the density at the surface of Jupiter will be to the mean density as 4 to 7, and consequently to the mean density of the earth as 94 ½ to 700. The earth is therefore not only more dense than Jupiter, but there is some ground to think, from what has been shown concerning those planets, that the ratio of the densities at their respective surfaces is greater than the ratio of their mean densities (or that of 94 ½ to 400), and that it approaches more towards the ratio of the densities of the rays of the sun incident upon them at their respective distances.

685. It cannot be expected that we should be able to discover, by observation, the variation of the distances of the satellites from *Kepler*'s law mentioned in art. 659. For let z denote the distance of the first satellite as it is determined by this law, from its periodic time, and from the distance and periodic time of the fourth satellite; that is, let the square of the periodic time of the fourth satellite be to the square of the periodic time of the first, as the cube of the distance of the fourth to z^3; let c denote the distance of the *focus* of the meridian section of Jupiter

piter from the centre; and the mean distance of the first satellite will be nearly $z + \frac{n+5}{n+3} \times \frac{cc}{15z}$; which, when the density is uniform, exceeds z by $\frac{cc}{10z}$ only, that is, by less than $\frac{1}{370th}$ part of Jupiter's semidiameter; and this excess is still less when n is greater than 1, or when the density is supposed to increase towards the centre. It would seem, therefore, that if there are any irregularities observed in the motions of those satellites, or indeed in any of the celestial motions, they are not to be ascribed to the consequences of the variation of the figure of the sun or planets from that of perfect spheres, but to their gravitation towards one another, or to some other causes.

686. We are next to apply the proposition demonstrated above from art. 636 to art. 641 to the theory of the tides. It follows from it, that if we suppose the earth to be fluid, and abstract from its motion on its axis, and the inclination of the right lines in which its particles gravitate towards the sun or moon, the figure which it would assume, in consequence of the unequal gravitation of its particles towards either of those bodies, would be accurately that of an oblong spheroid having its axis directed towards that body. For (*fig.* 296) let ADBE be any section of the earth through the right line DCE that is supposed to be directed towards the sun at S; and what was shown concerning the inequality of the gravities of the earth and moon towards the sun in art. 471 and 472. being applied to the particles of the earth, it will appear, in the same manner, that any particle P may be conceived to be affected by two forces, besides its gravity towards the earth; a force in the direction PC which the action of the sun adds to the gravitation of the particle P; and another in the direction Pk, parallel to CS, by which the action of the sun draws the particle from the plane AdB perpendicular to the right line SC at C. The former force is always as the distance PC; and if V represent this force at the mean distance d, then (PN and PM being perpendicular to AB and DE in N and M respectively) it may be resolved into a force PN $\times \frac{V}{d}$,

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perpendicular to the plane AdB and a force $PM \times \frac{V}{d}$ perpendicular to DE, which we now suppose to be the axis of this oblong spheroid. The latter force is $PN \times \frac{3V}{d}$. Therefore if the gravity at D be represented by D, and the gravity at A by A, CA and CD by a and b, as formerly; the particle P will gravitate in the direction PN perpendicular to the plane AdB with a force $\frac{D}{b} \times PN$ by art. 634, and the whole force, with which it will tend in that direction in consequence of its gravity and the other two forces, will be $\overline{\frac{D}{b} - \frac{2V}{d}} \times PN$. The particle tends in the direction PM perpendicular to the axis with a force $\overline{\frac{A}{a} + \frac{V}{d}} \times PM$. The former force is always as PN the distance of the particle P from the plane AdB to which its direction is perpendicular; and the latter as the distance from the axis DE. Therefore by art. 640, if the whole force at A be to the whole force at D (that is, if $A + \frac{2aV}{d}$ be to $D - \frac{2bV}{d}$) as b to a; the fluid will be every where in *æquilibrio*. And any particle P will tend towards the spheroid in a direction PK perpendicular to its surface APDB, with a force that is always measured by the right line PK terminated by the axis DE in K.

687. Let L represent the logarithm of the ratio of CA to DF, or of the subduplicate ratio of $b + c$ to $b - c$, the *modulus* being b; and by art. 647, D will be to A as $2abL - 2abc$ to $bbc - aaL$; consequently Db will be to Aa as $2bbL - 2bbc$ to $bbc - aaL$. Therefore if $L - c$ be represented by K, $Db - Aa$ will be to Db as $3bbK - ccK - ccc$ to $2bbK$, or (because K is equal to $\frac{c^3}{3b^2} + \frac{c^5}{5b^4} + \frac{c^7}{7b^6}$, &c.) as $\frac{2c^2}{5b^2} + \frac{12c^4}{35b^4}$, &c. to $1 + \frac{3c^2}{5b^2}$, &c. And (because $Db - Aa$ is equal to $\frac{2bb+aa}{d} \times V$, by the last article) $\frac{2bb+aa}{d} \times V$ is to Db in the same ratio. Hence if we suppose b equal to $d + x$, and a equal to $d - x$, we shall find that

that x is to d nearly as $15V$ to $8D$, or, more nearly, as $15V$ is to $8D - 9\frac{4}{7}V$; and that the excess of CD above CA is to the mean semidiameter d, as $15V$ to $4D - 4\frac{11}{14} \times V$.

688. The mean force, which the solar action adds to the gravity of the moon in the quadratures, is to the gravity of the moon towards the earth at her mean distance, in the duplicate ratio of the periodic time in which the moon would revolve about the earth in a circle at her mean distance, by her gravity towards the earth only, to the periodic time of the earth about the sun. By diminishing the former of these forces in the ratio of the mean distance of the moon to the semidiameter of the earth, and increasing the latter force in the duplicate ratio, Sir *Isaac Newton* finds V to be to D as 1 to 38604600. Therefore the ascent of the water under the equator, in consequence of its unequal gravitation towards the sun, ought to be to the semidiameter of the equator as 15 to 4×38604600; and this ascent ought to be about 1 foot $11\frac{1}{30}$ inches; which almost coincides with that which Sir *Isaac* found, by computing it briefly from what he had shown before concerning the figure of the earth. He deduces the lunar force from the solar, by comparing their effects in the syzygies, when they conspire together, with their effects in the quadratures of the sun and moon when these forces act against one other. The effect of the moon is much greater than of the sun, by common experience; and by his computations, the lunar force is to the solar as 448 to 100. These effects (according to observations and his theory) depend upon the positions of the luminaries to one another, their distances from the earth, their declinations from the equator, the latitudes of places, and the form and situation of the channels by which the tides are propagated to them from the ocean.

689. The ascent of the water, which was determined in the last article, is that which would be produced under the equator in consequence of the solar force, if the earth was fluid, and had no diurnal rotation; the gravitation towards the particles of the earth being supposed to decrease as the squares of the distances from them increase. But it does not follow, that the ascent of the water which arises from the solar action will be so great, if the oblong spheroid ADBE be in a different situation, and its

its transverse axis be not directed towards the sun; or when the whole mass (because of the constant figure of the solid parts) cannot assume the figure of such a spheroid. For the difference of CD and CA, that we have computed, proceeds not from the action of the sun only, but in part from the excess of the gravity at A above the gravity at D, which is owing to the spheroidical figure, and depends upon it. If the gravity had been supposed uniform in all parts of the surface, the ascent of the water would have not been above $\frac{3V}{D} \times d$, which is less than $\frac{15V}{4D} \times d$ by $\frac{3V}{4D}$, or one fifth part of the whole ascent. When the transverse axis of the oblong spheroid is directed towards the sun, the solar force and the diminution of gravity at the extremity of the transverse axis conspire together to produce the ascent of the water from A to D. But when DE the transverse axis of this oblong spheroid constitutes an angle with the right line CS, that joins the centres of the sun and earth, while the solar force endeavours to raise the water in this right line, the excess of the gravity at A above the gravity at D tends to raise it in a different part; and if by increasing the velocity of the diurnal rotation, the transverse axis DE should become perpendicular to CS, these causes would act directly against one another.

690. Sir *Isaac Newton* has shown that the lunar orbit (abstracting from its excentricity) ought to be an elliptic figure, having its centre in the centre of the earth, and the shorter axis directed to the sun, in consequence of the inequality of the gravity of the moon and earth towards the sun; and, supposing it to be a perfect ellipsis, endeavours to determine the ratio of the second axis to the transverse, *prop.* 28, *lib.* 3, *Princip.* In the same manner, if we should suppose the earth to revolve on its axis with a sufficient velocity, the particles of the sea at the equator would describe figures of an elliptic form about the centre of the earth, and revolve as satellites, without gravitating on those beneath them; and DE the greater axis of those figures being perpendicular to CS, the greatest ascent of the water would be at D and E. If we (*fig.* 297) should suppose all the sections of the earth perpendicular to its axis to be ellipses of this kind

similar

similar to each other; and the whole mass to form either an oblate spheroid, such as would be generated by the semi-ellipsis ADB revolving about the second axis AB, or an oblong spheroid, such as would be generated by DAE, about the transverse axis DE; then if the ratio of CD to CA was such, that (A and D being supposed to represent the gravities at A and D, as formerly) $A - \frac{CA}{CD} \times D$ should be to 3V as CA is to the mean semidiameter, the whole force that would act on each particle P, resulting from its gravity and the solar action, would be directed precisely to the centre C, and vary in the same ratio as the distance PC. For CD being represented by b, CA by a, and the mean semidiameter by d, as formerly, let PN be perpendicular to DE in N, PM perpendicular to AB in M, and Nq be taken upon NC in the same ratio to NC as $D \times a$ to $A \times b$, join Pq, and produce it till it meet AB in Q. Then, by art. 635, the gravity at P towards this spheroid will act in the direction PQ, and be always as PQ. Because Nq is to NC as Da to Ab, Cq is to NC, or CQ to MQ, as Ab — Da to Ab, or (by the supposition) as $\frac{3aV}{d}$ to A; that is, as the force by which the solar action endeavours to draw the water at A from the plane perpendicular to CS, to the gravity at A; or as the force Pk by which the sun endeavours to draw the particle P from that plane to the gravity at P in the direction PN, by art. 634. Therefore CQ is to PQ as the force Pk to the gravity at P; and the force which acts at P, compounded from the gravity and the force Pk, acts precisely in the direction PC, and varies in the same proportion as the distance PC. The other force which the solar action adds to the gravity is directed to C, and varies likewise as PC. Therefore the whole force that in this case acts on any particle P tends precisely to the centre of the spheroid, and is as the distance PC. And (by article 445) any particle P in the plane of the equator issuing from any point P with a just velocity, would describe the ellipse ADBE accurately, in the same time that a body would describe a circle about C at the distance DC by the force $D + \frac{b}{d} \times V$, or at any distance

distance CP by the whole force that acts at P: or if the earth was supposed to revolve on its axis in this time, the water in the canal EADB would move freely in this figure without gravitating on the bottom of the canal.

691. The ascent of the water in this case, or the excess of CD above CA, depends on the supposition that $Ab - aD$ is to $3Vb$ as a to d, by which the whole compounded force that acts on any particle of the spheroid is reduced precisely to the direction PC, so as to be measured by PC. To determine this ascent, and the form of the ellipsis EADB, the distance of the *focus* from the centre being represented by c, A was to D as $1 + \frac{2cc}{5bb}$, &c. to $1 + \frac{3cc}{10bb}$, &c. by art. 655, or (b being represented by $d + x$, and a by $d - x$) as $d + \frac{8x}{5}$ to $d + \frac{6x}{5}$ nearly; consequently, Ab is to Da as $d + \frac{13x}{5}$ to $d + \frac{x}{5}$; and $Ab - Da$ to Da as $\frac{12x}{5}$ to d. Therefore $3V$ is to D as $12x$ to $5d$, or x to d as $5V$ to $4D$; and the whole ascent of the water, or $2x$, to d as $5V$ to $2D$. This will be found to be the ascent of the water likewise, when the figure is supposed to be such an oblong spheroid as would be generated by a semi-ellipsis EAD about the axis ED. But when we abstracted from the diurnal rotation of the earth in art. 688, the ascent of the water was found to be $\frac{15V}{4D} \times d$ (the same which Sir *Isaac Newton* has defined) which is greater than this ascent $\frac{5V}{2D} \times d$ in the ratio of 3 to 2. The transverse axis of the spheroid DE is not in the position we have here supposed; but when the axis DE is inclined to CS in any angle, the ascent determined in art. 688 is to be diminished on this account.

692. Sir *Isaac Newton* having considered the ascent of the water, and the elliptic figure of the lunar orbit (abstracting from its excentricity) as similar *phænomena* arising from the solar force, let us imagine DBAE now to represent the lunar orb replenished with water; and the difference of the semidiameters

ters CD and CA, according to the last article, would be to the mean semidiameter as 5V to 2D, that is (in the present supposition) as 5 to 2 × 178 $\frac{29}{40}$, or as 1 to 71; and CA would be to CD as 70 to 71. According to *prop.* 28, *lib.* 3, *Princip.* this ratio is that of 69 to 70. This agreement seems to be accidental; but it appears from it, that if we had determined the ascent of the water, or the difference of CD and CA from the figure which is there ascribed to the lunar orbit, it would have been found nearly the same as in the last article. It would be found however nearly equal to that which was computed in art. 688, if we were to determine it from the figure which Dr. *Halley* ascribes to the lunar orbit from observations.

693. Because the earth is not fluid, and the solid parts retain the same figure in all positions of the sun or moon, the ascent of the water will be different from what was determined in art. 688, on this account likewise; and that the effect of this may be sensible, appears from art. 689, where we found that a difference of two feet only betwixt CD and CA, the semidiameters of the spheroid, gave occasion to an ascent of near five inches. If the earth was a solid sphere of an uniform density, and (abstracting from its diurnal rotation) the water in a small canal ADBE at the surface of its equator was affected by the solar force, it will be found (as in art. 492), that if we should suppose the water in the canal to assume such a figure, that the whole force which acts on any particle P resulting from the gravity and solar action, should be always perpendicular to the surface of the fluid, the difference of CD and CA would be to the semidiameter CD as 3V to 2D; consequently in this case the ascent of the water would be only $\frac{3}{5}$ of that which was defined in art. 688. The forces which produce this phænomenon are very minute in comparison of the gravity of the water; and circumstances, that in other enquiries are safely neglected, may have a sensible effect upon it.

694. There are particular causes, besides these mentioned by Sir *Isaac Newton*, and in the last article, that interfere in producing the various phænomena of the tides. The inequality of the velocities with which bodies revolve, by the diurnal motion

motion about the axis of the earth, in different latitudes, may have some effect on the motions of the sea and air; and may contribute to occasion greater tides, than might be otherwise expected from the theory, especially if their course be not far from the meridian. A current that sets out directly towards the north, ought, on this account, to bend its course soon afterwards somewhat towards the east; if it set out towards the south, its course ought afterwards to incline towards the west; and with the change of direction it may in some cases acquire greater force. Several phænomena may perhaps be accounted for from this consideration. But we are not to enter farther into this enquiry in this place.

695. If there is an ocean in *Jupiter*, the tides may be very considerable when all or most of his satellites are in one right line; and it may be worth while to observe, whether the great and sudden changes, that are sometimes perceived by astronomers to happen on the surface of this planet, have any analogy with their conjunctions and oppositions. If the other secondary planets, as well as the moon, move on their axis so as to have nearly the same hemisphere turned always towards their primary planet, the tides in their seas (if they have any) will chiefly proceed from the variation of their distances from it, and such may be sufficient; whereas their tides would probably be too great if they revolved on their axes with a greater velocity.

696. In this chapter we have endeavoured to determine accurately some of the consequences of Sir *Isaac Newton*'s theory of gravity; being persuaded that, however obscure the cause of gravity may be, there is hardly any proposition in experimental philosophy established on better evidence, than that there is such a power in Nature, and that it observes the laws we have supposed. We have sometimes made use of the term *attraction*, as a convenient expression only, and because it served to distinguish the real gravity from the apparent; which last is often a combination of gravity and several other powers. Sir *Isaac Newton* has shown how to compute the attraction of bodies, when the particles are supposed to attract each other according to other laws. We shall only subjoin an easy proof of one proposition on this subject. Suppose that the attraction

of the particles of the cone PAEa (*fig.* 283) decreases in the same proportion as the cubes of the distances from them increase; and a particle at P will tend to the spherical surface MNm (that has its centre in P) with a force that is as this surface (or PM^2) directly, and PM^3 inversely; that is, with a force which is as PM inversely, or directly as MV the ordinate of the hyperbola KVI described betwixt the asymptotes PA and PH. Therefore the attraction of the *frustum* MNmAEa will be measured by the hyperbolic area MVIA bounded by the ordinates at A and M; and the attraction of the cone PMNm by the infinite hyperbolic area that is conceived to be formed betwixt the ordinate MV and asymptote PH. It follows, that if such a law could take place, the particle P would tend towards the least portion of matter in contact with it by a greater force than towards the greatest body at any distance how small soever from it. The same is easily shown from art 297, when the attraction of the particles decreases as any powers of the distances, higher than their cubes, increase. As such laws would be very improper for preserving the celestial bodies in their regular courses (by art. 447 and 448), so they would be very unfit for producing a just force, by which their several parts might be kept together. The true law of gravity is better adapted for those purposes. It is the chain that holds the parts of each in a proper union, that perpetuates the motions in the great system about the sun, the preserves the revolutions in the lesser systems, of which it is composed, nearly regular. Its inequalities, in some cases, have their use, as in the tides; and a remarkable geometrical simplicity is often found in the conclusions that are deduced from it; of which we have had several instances, as in art. 445, 446, 636, 686, and 690.

BOOK

BOOK II.

OF THE COMPUTATIONS IN THE METHOD OF FLUXIONS.

CHAP. I.

Of the Fluxions of Quantities considered abstractly, or as represented by general Characters in Algebra.

697. THE idea of a fluxion, as described in art. 10 and 11, after Sir *Isaac Newton*, seemed to be more immediately applicable to geometrical magnitudes, which we may very naturally conceive to be generated by motion, than to quantities considered abstractly, or as they are expressed by general symbols in algebra. For this reason we chiefly considered the fluxions of geometrical magnitudes in the first book; and most commonly gave demonstrations from geometry, because these are often preferred, as more satisfactory than algebraic computations. The evidence of the method had been disputed, and objections had been made to the number of symbols employed in it, as if they might serve to cover defects in the principles and demonstrations. In order to obviate any suspicions of this kind, we endeavour to describe it in a manner that might represent the theorems plainly and fully, without any particular signs or characters, that they might be subjected more easily to a fair examination.

698. But an important part of this doctrine still remains to be described. The improvements that have been made by it, either in geometry or in philosophy, are in great measure owing to the facility, conciseness, and great extent of the method of computation, or algebraic part. It is for the sake of these

advantages that so many symbols are employed in algebra, the number and complication of which (together with the greater care there has been taken in treating of geometry, after the excellent models left us by the antients), have contributed more to occasion the preference that is often ascribed to geometry, in respect of perspicuity and evidence, than any essential difference that can be supposed to be between them. It is a general kind of arithmetic; and this is what renders its usefulness so universal; nor can this be supposed to derogate from its evidence, if we have no ideas more clear or distinct than those of numbers, and often acquire more satisfactory and distinct knowledge from computations than from constructions. It may have been employed to cover, under a complication of symbols, abstruse doctrines, that could not bear the light so well in a plane geometrical form; but without doubt, obscurity may be avoided in this art as well as in geometry, by defining clearly the import and use of the symbols, and proceeding with care afterwards.

699. The use of the negative sign in algebra is attended with several consequences that at first sight are admitted with difficulty, and has sometimes given occasion to notions that seem to have no real foundation. It implies that the real value of the quantity represented by the letter to which it is prefixed is to be subtracted; and it serves, with the positive sign, to keep in view what elements or parts enter into the composition of quantities, and in what manner, whether as increments or decrements (that is, whether by addition or subtration), which is of the greatest use in this art. In consequence of this, it serves to express a quantity of an opposite quality to the positive, as a line in a contrary position, a motion with an opposite direction, or a centrifugal force in opposition to gravity; and thus often saves the trouble of distinguishing, and demonstrating separately, the various cases of propositions, and preserves their analogy in view. But as the proportion of lines depends on their magnitude only, without regard to their position; and motions and forces are said to be equal, or unequal in any given ratio, without regard to their directions; and in general the proportion of quantities relates to their magnitude only, without determining whether they are to be considered as increments

ments or decrements; so there is no ground to imagine any other proportion of $-b$ and $+a$ (or of -1 and 1) than that of the real magnitudes of the quantities represented by b and a, whether these quantities are in any particular case to be added or subtracted. It is the same thing to subtract a decrement as to add an equal increment, or to subtract $-b$ from $a-b$ as to add $+b$ to it; and because multiplying a quantity by a negative number implies only a repeated subtraction of it, the multiplying $-b$ by $-n$, is subtracting $-b$ as often as there are units in n; and is therefore equivalent to adding $+b$ so many times, or the same as adding $+nb$. But if we infer from this, that 1 is to $-n$ as $-b$ to nb, according to the rule that unit is to one of the factors as the other factor is to the product; there is no ground to imagine that there is any mystery in this, or any other meaning than that the real magnitudes represented by 1, n, b and nb are proportional. For that rule relates only to the magnitude of the factors and product, without determining whether any factor, or the product, is to be added or subtracted. But this likewise must be determined in algebraic computations; and this is the proper use of the rules concerning the signs, without which the operation could not proceed. Because a quantity to be subtracted is never produced, in composition, by any repeated addition of a positive, or repeated subtraction of a negative, a negative square number is never produced by composition from a root. Hence the $\sqrt{-1}$, or the square-root of a negative, implies an imaginary quantity, and in resolution is a mark or character of the impossible cases of a problem; unless it is compensated by another imaginary symbol or supposition, when the whole expression may have a real signification. Thus $1+\sqrt{-1}$, and $1-\sqrt{-1}$ taken separately are imaginary, but their sum is 2; as the conditions that separately would render the solution of a problem impossible, in some cases destory each other's effect when conjoined. In the pursuit of general conclusions, and of simple forms for representing them, expressions of this kind must sometimes arise where the imaginary symbol is compensated in a manner that is not always so obvious. By proper substitutions, however, the expression may be transformed into

quantity, wherein each particular term may have a real signification, as well as the whole expression. The theorems that are sometimes briefly discovered by the use of this symbol, may be demonstrated without it by the inverse operation, or some other way: and though such symbols are of some use in the computations in the method of fluxions, its evidence cannot be said to depend upon any arts of this kind. We have just mentioned these things without enlarging upon them, for we suppose that the common algebra is admitted.

700. The rules for the computations in this method may be deduced from art. 99. but it may be worth while to demonstrate them here briefly, from general principles that may seem more immediately applicable to algebraic quantities. Any quantities that are produced from each other by an algebraic operation, or whose relation is expressed by any algebraic form, being supposed to increase or decrease together, some will be found to increase or decrease by greater differences, or at a greater rate; others by less differences, or at a less rate; and while some are supposed to increase or decrease at one constant rate by equal successive differences, others increase or decrease by differences that are always varying. We have no occasion for considering such quantities in this doctrine as generated by motion, and for enquiring into the velocities of those motions, or for considering the prime or ultimate ratio of their increments or decrements, but for ascertaining the respective rates, according to which they increase or decrease, when they are supposed to vary together; in order from these to discover the properties of the quantities themselves. Thus by comparing the velocities of points that are supposed to generate lines at the same time, it appears when a line increases at a greater or less rate than another, and in what proportion. The same is to be said of any quantities, which, while they vary together, are always in the same proportion to one another as those lines. But it does not seem to be necessary to have always recourse to such suppositions; though in treating of geometrical magnitudes, that are often conceived to be generated by motion, this method of comparing the rates of their increase or decrease is natural and clear, and has other advantages *. When a quan-

* Art. 474.

tity A increases by differences equal to a, 2A increases or decreases by differences equal to $2a$, and manifestly increases or decreases at a greater rate than A in the proportion of $2a$ to a, or 2 to 1; and if m and n be invariable, $\frac{mA}{n}$ increases or decreases by differences equal to $\frac{ma}{n}$; and therefore at a greater or less rate than A in proportion as $\frac{ma}{n}$ is greater or less than a, or m is greater or less than n. This seems to be easily conceived, without having recourse to any other considerations than the relations of the differences by which the quantities increase or decrease. In order therefore to avoid the frequent repetition of figurative expressions in this algebraic part as much as possible, we will endeavour to substitute in place of the definitions and axioms above (art. 11 and 15), others that are rather of a more general import, but are perfectly consistent with them, and are best explained by them; as other principles and propositions in algebra are commonly best illustrated from geometry.

701. By the *fluxions* of quantities we shall therefore now understand, *any measures of their respective rates of increase or decrease, while they vary (or flow) together.* There can be no difficulty in determining these measures when the quantities increase or decrease by successive differences that are always in the same invariable proportion to each other, as in the last article. While A by increasing becomes equal to $A + a$, or by decreasing equal to $A - a$, 2A becomes equal to $2A + 2a$, or to $2A - 2a$; and as 2A increases or decreases at a greater rate than A in the proportion of $2a$ to a, so the fluxion of A being supposed equal to a, the fluxion of 2A must be equal to $2a$. In the same manner the fluxion of $\frac{m}{n} \times A$ (or of $\frac{m}{n} \times A \mp e$, supposing m, n, and e to be invariable), is $\frac{m}{n} \times a$; and since m may be to n in any assignable ratio, a quantity may be always assigned that shall increase or decrease at a greater or less rate than A in any proportion, or that shall have its flux-

ion greater or less than the fluxion of A in any ratio. In such cases the ratio of the fluxions and that of the differences by which the quantities increase or decrease are the same.

702. But while A is supposed to increase at a constant rate by any equal successive differences, if B increase or decrease by differences that are always varying, B cannot be said to increase or decrease at any one constant rate; and it is not so obvious how, the fluxion of A being supposed equal to its increment a, the variable fluxion of B is to be determined. It cannot be supposed that the fluxions and differences are always in the same proportion in this case; but it is evident, however, that if B increase by differences that are always greater than the equal successive differences by which $\frac{m}{n} \times A$ increases, it cannot increase at a less rate than $\frac{m}{n} \times A$; and that it cannot increase at a greater rate than $\frac{m}{n} \times A$, while its successive differences are always less than those of $\frac{m}{n} \times A$. The fluxion of A being still represented by a, the fluxion of B therefore cannot be less than $\frac{m}{n} \times a$ in the former case, or greater than $\frac{m}{n} \times a$ in the latter. The following propositions are consequences of this, and will enable us to determine at what rate B increases when its relation to A is known.

703. The successive values of the root A being represented by $A - a$, A, $A + a$, *&c.* which increase by any constant difference a, let the corresponding values of any quantity produced from A by any algebraic operation (or that has a dependance upon it so as to vary with it) be $B - b$, B, $B + b$, *&c.* Then if the successive differences b, b, *&c.* of the latter quantity always increase, how small soever a may be, then B cannot be said to increase at so great a rate as a quantity that increases uniformly by equal successive differences greater than b, or at so small a rate as any quantity that increases uniformly by equal successive differences less than b. In like manner, if the relation of the quantities is such, that the successive differences

ences, b, b, &c. continually decrease; then B cannot be said to increase at the same rate as a quantity that increases uniformly by equal successive differences greater than b, or less than b.

704. Therefore the fluxion of A being supposed equal to the increment a, the fluxion of B cannot be greater than b or less than b, when the successive differences b, b, &c. continually increase; and cannot be greater than b, or less than b, when these successive differences always decrease.

705. In the same manner if the latter quantity decrease while the former increases, and its successive values be B + b, B, B — b, &c. then if the decrements b, b, &c. continually increase, B cannot be said to decrease at so great a rate as a quantity that decreases uniformly by equal successive differences greater than b, or at so low a rate as a quantity that decreases uniformly by equal successive difference less than b. Therefore, in this case, the fluxion of A being supposed equal to a, the fluxion of B cannot be greater than b, or less than b. And in the same manner if the successive decrements b, b, &c. always decrease, the fluxion of B cannot be greater than b or less than b.

706. As the fluxions of quantities are any measures of the respective rates according to which they increase or decrease, by art. 701, so it is of no importance how great or small soever those measures are, if they be in the just proportion or relation to each other. Therefore if the fluxions of A and B may be supposed equal to a and b, respectively, they may be likewise supposed equal to $\frac{1}{2}a$ and $\frac{1}{2}b$, or to $\frac{ma}{n}$ and $\frac{mb}{n}$. These principles (as other algebraic propositions) may be illustrated from geometry, as we observed in art. 700. And the propositions concerning the fluxions of areas, ordinates, &c. in the first book, may be demonstrated immediately from them; but this would be needless.

707. Prop. I. *The fluxion of the root* A *being supposed equal to* a, *the fluxion of the square* AA *will be equal to* 2A × a.

Let the successive values of the root be A—u, A, A + u, and the corresponding values of the square will be AA — 2Au

$2Au + uu$, AA, $AA + 2Au + uu$, which increase by the differences $2Au - uu$, $2Au + uu$, &c. and because those differences increase, it follows from art. 704, that if the fluxion of A be represented by u, the fluxion of AA cannot be represented by a quantity that is greater than $2Au + uu$, or less than $2Au - uu$. This being premised, suppose, as in the proposition, that the fluxion of A is equal to a; and if the fluxion of AA be not equal to $2Aa$, let it first be greater than $2Aa$ in any ratio, as that of $2A + o$ to $2A$, and consequently equal to $2Aa + oa$. Suppose now that u is any increment of A less than o; and because a is to u as $2Aa + oa$ to $2Au + ou$, it follows (art. 706) that if the fluxion of A should be represented by u, the fluxion of AA would be represented by $2Au + ou$, which is greater than $2Au + uu$. But it was shown, from art. 704, that if the fluxion of A be represented by u, the fluxion of AA cannot be represented by a quantity greater than $2Au + uu$. And these being contradictory, it follows that the fluxion of A being equal to a, the fluxion of AA cannot be greater than $2Aa$. If it can be less than $2Aa$, when the fluxion of A is supposed equal to a, let it be less in any ratio of $2A - o$ to $2A$, and therefore equal to $2Aa - oa$. Then because a is to u as $2Aa - oa$ is to $2Au - ou$, which is less than $2Au - uu$ (u being supposed less than o, as before), it follows that if the fluxion of A was represented by u, the fluxion of AA would be represented by a quantity less than $2Au - uu$, against what has been shown from art. 704. Therefore, the fluxion of A being supposed equal to a, the fluxion of AA must be equal to $2Aa$.

708. The fluxions of A and B being supposed equal to a and b, respectively, the fluxion of $A + B$ will be $a + b$; the fluxion of $\overline{A+B}^2$ or of $AA + 2AB + BB$, will be $2 \times \overline{A+B} \times \overline{a+b}$ or $2Aa + 2Bb + 2Ba + 2Ab$, by the last article. The fluxion of $AA + BB$ is $2Aa + 2Bb$, by the same; consequently the fluxion of $2AB$ is $2Ba + 2Ab$; and the fluxion of AB is $Ba + Ab$. Hence if P be equal to AB, and the fluxion of P be p, then p will be equal to $Ba + Ab$, and dividing by P, or AB, we find $\frac{p}{P} = \frac{a}{A} + \frac{b}{B}$. If $Q = \frac{A}{B}$, and q be the fluxions of Q, then

then $QB = A, \frac{q}{Q} + \frac{b}{B} = \frac{a}{A}$ or $\frac{q}{Q} = \frac{a}{A} - \frac{b}{B}$; and consequently $q = \frac{Qa}{A} - \frac{Qb}{B} = \frac{a}{B} - \frac{Ab}{BB}$ or $\frac{aB - Ab}{BB}$. When any of the quantities decrease, its fluxion is to be considered as negative.

709. If n be any integer number, and the sum of the terms E^{n-1}, $E^{n-2}F$, $E^{n-3}F^2$, $E^{n-4}F^3$, &c. continued till their number be equal to n, be multiplied by $E - F$, the product will be $E^n - F^n$. For the terms being formed by subducting continually unit from the index of E and adding it to the index of F, the last term will be F^{n-1}. The product of their sum multiplied by E will be $E^n + E^{n-1}F + E^{n-2}F^2 \ldots + EF^{n-1}$; their sum multiplied by $-F$ gives $-E^{n-1}F - E^{n-2}F^2 \ldots - EF^{n-1} - F^n$; and the sum of these two products is $E^n - F^n$.

710. Supposing E to be greater than F, $E^n - F^n$ will be less than $nE^{n-1} \times \overline{E-F}$, but greater than $nF^{n-1} \times \overline{E-F}$. For each of the terms E^{n-1}, $E^{n-2}F$, $E^{n-3}F^2$, &c. is greater than the subsequent term in the same ratio that E is greater than F, and E^{n-1} is the greatest term; consequently the number of terms being equal to n, nE^{n-1} is greater than their sum; and $nE^{n-1} \times \overline{E-F}$ is greater than their sum multiplied by $E-F$, or (by the last article) greater than $E^n - F^n$. Because the last term F^{n-1} is less than any preceding term, $nF^{n-1} \times \overline{E-F}$ is less than the sum of the terms multiplied by $E - F$, or less than $E^n - F^n$.

711. When n is any integer positive number, the root A being supposed to increase by any equal successive differences, the successive differences of the power A^n will continually increase. For let $A - a$, A, $A + a$, be any successive values of the root, and $\overline{A-a}^n$, A^n, $\overline{A+a}^n$ will be the corresponding values of the power. But $\overline{A+a}^n - A^n$ is greater than $nA^{n-1}b$; as appears by substituting, in the last article, $A + a$ for E, A for F, and a for $E - F$. In like manner $nA^{n-1}a$ is greater than $A^n - \overline{A-a}^n$. Therefore $\overline{A+a}^n - A^n$ is greater than $A^n - \overline{A-a}^n$, and the successive differences of the power continually increase.

712. Prop.

advantages that so many symbols are employed in algebra, the number and complication of which (together with the greater care there has been taken in treating of geometry, after the excellent models left us by the antients), have contributed more to occasion the preference that is often ascribed to geometry, in respect of perspicuity and evidence, than any essential difference that can be supposed to be between them. It is a general kind of arithmetic; and this is what renders its usefulness so universal; nor can this be supposed to derogate from its evidence, if we have no ideas more clear or distinct than those of numbers, and often acquire more satisfactory and distinct knowledge from computations than from constructions. It may have been employed to cover, under a complication of symbols, abstruse doctrines, that could not bear the light so well in a plane geometrical form; but without doubt, obscurity may be avoided in this art as well as in geometry, by defining clearly the import and use of the symbols, and proceeding with care afterwards.

699. The use of the negative sign in algebra is attended with several consequences that at first sight are admitted with difficulty, and has sometimes given occasion to notions that seem to have no real foundation. It implies that the real value of the quantity represented by the letter to which it is prefixed is to be subtracted; and it serves, with the positive sign, to keep in view what elements or parts enter into the composition of quantities, and in what manner, whether as increments or decrements (that is, whether by addition or subtration), which is of the greatest use in this art. In consequence of this, it serves to express a quantity of an opposite quality to the positive, as a line in a contrary position, a motion with an opposite direction, or a centrifugal force in opposition to gravity; and thus often saves the trouble of distinguishing, and demonstrating separately, the various cases of propositions, and preserves their analogy in view. But as the proportion of lines depends on their magnitude only, without regard to their position; and motions and forces are said to be equal, or unequal in any given ratio, without regard to their directions; and in general the proportion of quantities relates to their magnitude only, without determining whether they are to be considered as increments

ments or decrements; so there is no ground to imagine any other proportion of $-b$ and $+a$ (or of -1 and 1) than that of the real magnitudes of the quantities represented by b and a, whether these quantities are in any particular case to be added or subtracted. It is the same thing to subtract a decrement as to add an equal increment, or to subtract $-b$ from $a-b$ as to add $+b$ to it; and because multiplying a quantity by a negative number implies only a repeated subtraction of it, the multiplying $-b$ by $-n$, is subtracting $-b$ as often as there are units in n; and is therefore equivalent to adding $+b$ so many times, or the same as adding $+nb$. But if we infer from this, that 1 is to $-n$ as $-b$ to nb, according to the rule that unit is to one of the factors as the other factor is to the product; there is no ground to imagine that there is any mystery in this, or any other meaning than that the real magnitudes represented by 1, n, b and nb are proportional. For that rule relates only to the magnitude of the factors and product, without determining whether any factor, or the product, is to be added or subtracted. But this likewise must be determined in algebraic computations; and this is the proper use of the rules concerning the signs, without which the operation could not proceed. Because a quantity to be subtracted is never produced, in composition, by any repeated addition of a positive, or repeated subtraction of a negative, a negative square number is never produced by composition from a root. Hence the $\sqrt{-1}$, or the square-root of a negative, implies an imaginary quantity, and in resolution is a mark or character of the impossible cases of a problem; unless it is compensated by another imaginary symbol or supposition, when the whole expression may have a real signification. Thus $1+\sqrt{-1}$, and $1-\sqrt{-1}$ taken separately are imaginary, but their sum is 2; as the conditions that separately would render the solution of a problem impossible, in some cases destory each other's effect when conjoined. In the pursuit of general conclusions, and of simple forms for representing them, expressions of this kind must sometimes arise where the imaginary symbol is compensated in a manner that is not always so obvious. By proper substitutions, however, the expression may be transformed into

 another,

712. Prop. II. *The fluxion of the root A being supposed equal to a, the fluxion of the power A^n will be naA^{n-1}.*

For if the fluxion of A^n can be greater than naA^{n-1}, let the excess be equal to any quantity r; suppose o equal to the excess of $\sqrt[n-1]{A^{n-1}+\frac{r}{na}}$ above A, and consequently $\overline{A+o}^{n-1} = A^{n-1}+\frac{r}{na}$. Then $na\times\overline{A+o}^{n-1}$ will be equal to $naA^{n-1}+r$, the fluxion of A^n. Let u be any increment of A less than o; and because a is to u as $na\times\overline{A+o}^{n-1}$ to $nu\times\overline{A+o}^{n-1}$, it follows (by art. 706), that if the fluxion of A be now represented by the increment u, the fluxion of A will be represented by $nu\times\overline{A+o}^{n-1}$ which is greater than $nu\times\overline{A+u}^{n-1}$, and this last is itself greater than $\overline{A+u}^{n-1}-A^n$, by art. 710. But when the successive values of the root are $A-u$, A, $A+u$, those of the power are $\overline{A+u}^n$, A^n, $\overline{A+u}^n$, the successive differences of which continually increase; consequently (by art. 704), if the fluxion of A be represented by u, the fluxion of A^n cannot be represented by a quantity greater than $\overline{A+u}^n-A^n$, or less than $A^n-\overline{A+u}^n$. And these being contradictory, it follows that when the fluxion of A is supposed equal to a, the fluxion of A^n cannot be greater than naA^{n-1}. If it can be less than naA^{n-1}, let it be equal to $naA^n - r$, or (by supposing $o = A-\sqrt[n-1]{A^{n-1}-\frac{r}{na}}$) to $na\times\overline{A-o}^{n-1}$. Then u being supposed less than o, if the fluxion of A was represented by u, the fluxion of A^n would be represented by $nu\times\overline{A-o}^{n-1}$, which is less than $nu\times\overline{A-u}^{n-1}$ (because we suppose u to be less than o) and therefore less than $A^n-\overline{A-u}^n$ by art. 710. But this is repugnant to what has been demonstrated from art. 704. Therefore the fluxion of A being supposed equal to a, the fluxion of A^n must be equal to naA^{n-1}.

713. The

713. The fluxions of $\frac{1}{A^n}$, or of $A^{\frac{1}{n}}$, may be determined in the same manner: but these being comprehended in the following theorem, it is needless to consider them separately. We shall only observe that the *lemma* for determining the former is, that when E is greater than F, $\frac{1}{F^n} - \frac{1}{E^n}$ or $\frac{E^n - F^n}{E^nF^n}$ is less than $\frac{nE^{n-1}}{E^nF^n} \times \overline{E-F}$ (by art. 710), or $\frac{nE-nF}{EF^n} \times \overline{E-F}$ which is less than $\frac{nE-nF}{F^{n+1}}$; but greater than $\frac{nF^{n-1}}{E^nF^n} \times \overline{E-F}$ (art. 710), and consequently greater than $\frac{nE-nF}{E^{n+1}}$. And hence it may be demonstrated, as in art. 712, that when the fluxion of A is supposed equal to a, the fluxion of $\frac{1}{A}$ is $\frac{-na}{A^{n+1}}$, the sign being negative because $\frac{1}{A^n}$ decreases while A increases. We have supposed n to be an integer positive number in this and the last article.

714. Prop. III. *The fluxion of A being supposed equal to a, the fluxion of $\overline{A}^{\frac{m}{n}}$ will be $\frac{ma}{n} \times \overline{A}^{\frac{m}{n}-1}$.*

First, let the exponent $\frac{m}{n}$ be any positive fraction whatsoever; suppose $\overline{A}^{\frac{m}{n}} = K$; consequently $\overset{m}{A} = \overset{n}{K}$; and the fluxion of K being supposed equal to k, $maA^{m-1} = nkK^{n-1}$, by art. 712, and

and k or the fluxion of $A^{\frac{m}{n}}$ will be equal to $\frac{maA^{m-1}}{nK^{n-1}} = \frac{maK}{nA} = \frac{m}{n} \times aA^{\frac{m}{n}-1}$. When $\frac{m}{n}$ is negative, let it be equal to $-r$; and suppose $A^{-r} = K$, or $1 = A^rK$; then taking the fluxions (by art. 708), $rA^{r-1}aK + kA^r = o$, and $k = -\frac{rA^{r-1}aK}{A^r} = -rA^{-r-1}a = \frac{m}{n} \times A^{\frac{m}{n}-1}a$.

715. Prop. IV. *Suppose* P *to be the product of any factors* A, B, C, D, E, *&c.* (*or* P = ABCDE, *&c.*) *let the fluxions of* P, A, B, C, D, E, *be respectively equal to* $p, a, b, c, d, e,$ *&c.* *and* $\frac{p}{P}$ *will be equal to* $\frac{a}{A} + \frac{b}{B} + \frac{c}{C} + \frac{d}{D}$ *&c.*

Let Q be equal to the product of all the factors of P, the first A excepted; that is, suppose P = AQ. Suppose R equal to the product of all the factors, the first two A and B excepted, that is, let P = ABR, or Q = BR. In the same manner let R = CS, S = DT, and so on. Then, the fluxions of Q, R, S, T, &c. being supposed respectively equal to $q, r, s, t,$ &c. it follows, from art. 708, that $\frac{p}{P} = \frac{a}{A} + \frac{q}{Q} =$ (because $\frac{q}{Q} = \frac{b}{B} + \frac{r}{R}$) $\frac{a}{A} + \frac{b}{B} + \frac{r}{R} =$ (because $\frac{r}{R} = \frac{c}{C} + \frac{s}{S}$) $\frac{a}{A} + \frac{b}{B} + \frac{c}{C} + \frac{s}{S}$ = (because $\frac{s}{S} = \frac{d}{D} + \frac{t}{T}$) $\frac{a}{A} + \frac{b}{B} + \frac{c}{C} + \frac{d}{D} + \frac{t}{T}$, and so on. Therefore $\frac{p}{P}$ is equal to the sum of the quotients when the fluxion of each factor of P is divided by the factor itself.

716. If the factors be supposed equal to each other, and their number be equal to n, then P = A^n, and by the last proposition

position $\frac{p}{P}=\frac{na}{A}$; consequently $p=\frac{nPa}{A}=naA^{n-1}$; as we found in art. 712.

717. Prop. V. *If* $P=\frac{ABC\times \&c.}{KLM\times \&c.}$ *and the fluxions of the respective quantities be expressed by the small letters, p, a, b, c,* &c. *as formerly, then* $\frac{p}{P}=\frac{a}{A}+\frac{b}{B}+\frac{c}{C}-\frac{k}{K}-\frac{l}{L}-\frac{m}{M}$, *&c.*

For PKLM × *&c.* = ABC × *&c.* and, by art. 715, $\frac{p}{P}+\frac{k}{K}+\frac{l}{L}+\frac{m}{M}$, *&c.* $=\frac{a}{A}+\frac{b}{B}+\frac{c}{C}$, *&c.* whence by transposition $\frac{p}{P}=\frac{a}{A}+\frac{b}{B}+\frac{c}{C}-\frac{k}{K}-\frac{l}{L}$, *&c.*

718. The fluxion of the logarithms being supposed invariable, the fluxions of any quantities N and M will be in the same proportion as these quantities themselves. For it is the fundamental property of the logarithms, that when they are taken in any arithmetical progression, the quantities of which they are the logarithms are always in a geometrical progression. Therefore, the logarithms being supposed to increase by any equal differences, these quantities will increase or decrease by differences that increase or decrease in the same proportion as the quantities themselves. Let A — *a*, A, A + *a*, be the respective logarithms of N — n, N, N + *n*; and B — *a*, B, B + *a* the logarithms of M — m M, M + *m*; then because the logarithms increase by the constant difference *a*, n will be to *n* as N to N + *n*; m to *m* as M to M + *m*; and *n* to m as N + *n* to M. Therefore when the quantities and their logarithms increase together, it follows from art. 704, that if the constant fluxion of the logarithm be supposed equal to its increment *a*, the fluxion of N will not be greater than *n*, or the fluxion of M less than m; consequently the fluxion of N is to the fluxion of N in a ratio that is not greater than that of *n* to m, or of N + *n* to M. But if the fluxion of N could be to the fluxion

fluxion of M in any ratio greater than that of N to M, as in that of $N+u$ to M; then by supposing n to be less than u, the fluxion of N would be to the fluxion of M in a ratio greater than that of $N+n$ to M. And these being contradictory, it follows that the ratio of these fluxions is not greater than that of N to M. In the same manner the fluxion of M is to the fluxion of N in a ratio that is not greater than that of M to N. Therefore the ratio of the fluxions of M and N is the same with the ratio of the quantities M and N. When the quantities decrease while the logarithms increase, the demonstration is the same.

719. Prop. VI. *The fluxion of any quantity* N *is to the fluxion of its logarithm as* N *is to the* modulus *of the logarithmic system.*

For the quantities and their logarithms being supposed to increase or decrease together, when the quantity increases or decreases at the same rate as its logarithm, it is then equal to the *modulus*. Suppose this quantity to be M, and since the fluxion of N is to the fluxion of M as N is to M, by the last article; it follows that the fluxion of N is to the fluxion of its logarithm as N is to the *modulus*. Hence if $N = A^e$, e being any invariable exponent, the log. $N = e \times \log. A$, consequently, the fluxions of N and A being supposed equal to n and a respectively, $\frac{Mn}{N} = \frac{eMa}{A}$, and $n = \frac{eNa}{A} = eA^{e-1}a$. We insisted on this, at some length, in chap. 6. book I.

720. When the fluxion of a quantity is variable, it may be considered as a fluent; and its fluxion may be determined (which is called the second fluxion of that quantity) by the preceding propositions. Thus we found in art. 707, that the fluxion of A being supposed equal to a, the fluxion of AA is $2Aa$; and if A be supposed to increase at an uniform rate, or its fluxion a be invariable, $2Aa$ will increase by equal successive differences; consequently its fluxion, or the second fluxion of AA, will be equal to any of those differences (art. 701), as to $2a \times \overline{A+a} - 2Aa$, or $2aa$. If a be variable, let its fluxion be equal

equal to z, and the fluxion of $2Aa$ (or second fluxion of AA) will be $2aa + 2Az$, by art. 708. In the same manner, the fluxion of A being constant, the fluxion of $nA^{n-1}a$, or the second fluxion of A^n, is $na \times \overline{n-1} \times A^{n-2}a$, or $n \times \overline{n-1} \times aa\, A^{n-2}$; the fluxion of this, or the third fluxion of A^n, is $n \times \overline{n-1} \times \overline{n-2} \times a^3A^{n-3}$. And the fluxion of A^n of any order denoted by m, is $n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}$, &c. $\times a^mA^{n-m}$, where the factors in the coefficient are to be continued till their number be equal to m. When n is any integer positive number, the fluxion of A^n, of the order n, is invariable and equal to $n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}$, &c. $\times a^n$. The quantities that represent those fluxions of A^n depend on a, which represents the fluxion of A. When A remains of the same value, the first fluxion A^n is greater or less in the same proportion as a is supposed to be greater or less; the second fluxion of A^n is in the duplicate ratio of a; and its fluxion of the order m is as a^m. If a be variable, but z the fluxion of a, or the second fluxion of A, be constant, then the fourth fluxion of AA will be constant and equal to $6zz$; for we found that the second fluxion of AA was $2aa + 2Az$; the fluxion of which is $4az + 2az$, or $6az$; and the fluxion of this is $6zz$. In like manner the sixth fluxion of A^3 will be constant in this case, and equal to $90z^3$.

721. The second differences of any quantity B are the successive differences of its first differences; and as the fluxion of B increases when its successive differences increase, so its second fluxion, or its fluxions of any higher order, increase, when its second or higher differences increase. If we arrive at differences of any order that are constant, the fluxion of the same order is constant, and is expressed by that difference. Thus when A is supposed to increase by constant differences equal to a, and its fluxion is supposed equal to a, the second difference of AA

(or $\overline{A+a}^2 - 2AA + \overline{A-a}^2$) is $2aa$, which is likewise its second fluxion; and the third difference of A^3 is $6a$, which is its third fluxion. When n is any integer and positive number, the fluxion of A^n of the order n is equal to the fluxion of any of its first differences of the next inferior order, or to the fluxion of any of its second differences of the order $n-2$, and so on. For the fluxion of $\overline{A+a}^n - A^n$ (one of the first differences of A^n) of the order $n-1$ is $n \times \overline{n-1} \times \overline{n-2}$, &c. $\times \overline{A+a}^{n-n+1} - A \times a^{n-1} = n \times \overline{n-1} \times \overline{n-2}$, &c. $\times a^n$, where the coefficients are supposed to be continued till their number be $n-1$, so that the last must be 2. And this we found to be the fluxion of A^n of the order n, in the preceding article. In the same manner, the fluxion of $\overline{A+a}^n - 2A^n + \overline{A-a}^n$, (the second difference of A^n) of the order $n-2$, is equal to the fluxion of $\overline{A+a}^n - A$ of the order $n-1$; and consequently equal to the fluxion of A^n of the order n. These fluxions are invariable and equal to the last or invariable differences. But in other cases the fluxions of A^n of any order are less than its subsequent differences of the same order, but greater than the preceding differences, as in art. 703.

722. The preceding propositions are demonstrated briefly by finding the ultimate relation of the differences of the fluents, for this will determine their respective rates of increasing or decreasing, or the relation of their fluxions. Thus, because $\overline{A+a}^n - A^n$, the increment of A^n, is less than $na \times \overline{A+a}^{n-1}$, but greater than naA^{n-1}, by art. 710; and when a is supposed to be diminished continually till it vanish, the ultimate ratio of $na \times \overline{A+a}^{n-1}$ to naA^{n-1} is a ratio of equality: it follows that the ultimate ratio of the increment $\overline{A+a}^n - A^n$ to naA^{n-1} is a ratio of equality; and that the fluxion of A being supposed equal to a, the fluxion of A^n must be $na\,A^{n-1}$

as

as in art. 712. In the same manner the second or higher fluxions of A^n, or of any other fluent, are ultimately equal to the corresponding differences of the fluent. If we suppose (with Mr. *Leibnitz,* and those who have followed his method) a to be an infinitely small difference of A, and suppose quantities to be equal when their difference is infinitely less than the quantities themselves, $na \times \overline{A+a}^{n-1}$ must be supposed equal to naA^{n-1}; and, since $\overline{A+a}^n - A^n$, the difference of A^n, cannot be greater than the former, or less than the latter (art. 710), it must be supposed equal to naA^{n-1}.

CHAP. II.

Of the Notation of Fluxions, the Rules of the direct Method, and the fundamental Rules of the inverse Method of Fluxions.

723. SIR *Isaac Newton,* on some occasions,* represented the fluents by capital letters, and their fluxions by the small letters that correspond to them. We followed this notation in the last chapter, in demonstrating the grounds of the operations. But it is convenient that the fluxions should be distinguished from other algebraic expressions, and in such a manner that the second and higher fluxions may be represented so as to preserve the original fluent in view. In his last method he represented the variable or flowing quantities by the final letters of the alphabet, as x, y, z; their first fluxions by the same letters pointed once, as by $\dot{x}, \dot{y}, \dot{z}$; their second fluxions by the same letters pointed twice, as by $\ddot{x}, \ddot{y}, \ddot{z}$; the third fluxions by the letters pointed thrice, as by $\dddot{x}, \dddot{y}, \dddot{z}$, and so on, where the number of points serves to show the order of the fluxion that is represented with respect to the first fluent; and the difference of those numbers show of what order any of

* Princip. lib. ii. lemm. 2.

 them

them is the fluxion of those that precede it, as $\dot{\ddot{y}}$ is the first fluxion of $\ddot{y}$, but the second fluxion of $\dot{y}$. Mr. *Leibnitz* represented the infinitely small differences of x, y, z, by dx, dy, dz; their second differences by ddx, ddy, ddz; and their infinitesimal differences of any order n, by $d^n x, d^n y, d^n z$. The symbol $\dot{x}$, or dx, expresses the fluxion of x generally, without determining whether it is to be considered as positive or negative; that is, whether x increases or decreases with respect to the other fluents. Invariable quantities are represented by the first letters of the alphabet, as a, b, c, &c. These have no fluxions; and, in the same manner, when any fluxion is supposed constant, its fluxion vanishes. Sir *Isaac Newton** has comprehended most of the rules of the direct method in one general proposition; but it is more usual to represent them separately; and it may be of use to proceed gradually from the simple cases to those that are more complex.

724. I. When one simple fluent only enters each term of a compound quantity, the fluxion of this quantity is found by collecting the fluxions of each term, or by placing a point over each fluent. Thus the fluxion of $x+y-z$, is $\dot{x}+\dot{y}-\dot{z}$; the fluxion of $ax+by-cz$ is $a\dot{x}+b\dot{y}-c\dot{z}$. The fluxion of ax, or of $ax+bb$, is $a\dot{x}$. This rule is obvious, and follows from art. 701, or art. 36, 41, and 78.

725. II. As the fluxion of xy is $\dot{x}y+\dot{y}x$ (by art. 708 and 99), so the fluxion of a product of any two fluents is the sum of the several products when the fluxion of each factor is multiplied by the other factor. Thus the fluxion of $\overline{a+x} \times \overline{b-y}$ is $\dot{x} \times \overline{b-y} - \dot{y} \times \overline{a+x} = \dot{x}b - \dot{x}y - \dot{y}a - \dot{y}x$. As the fluxion of az is $a\dot{z}$; so the fluxion of axy is $a \times \overline{\dot{x}y+\dot{y}x} = \dot{x}ya + \dot{y}xa$.

726. III. As the fluxion of the fraction $\frac{x}{y}$ is $\frac{\dot{x}y-\dot{y}x}{yy}$ by art. 708, so the fluxion of any fraction is found by multiplying the fluxion of the numerator by the denominator, substracting the

* See his Lemma II. to Prop. VIII. lib. II. of his Principia.

product

product of the fluxion of the denominator multiplied by the numerator, and dividing the remainder of the square of the denominator. Thus the fluxion of $\frac{a-x}{a+x}$ is $\frac{-\dot{x}\times\overline{a+x}-\dot{x}\times\overline{a-x}}{\overline{a+x}^2}$ $=\frac{-2a\dot{x}}{aa+2ax+xx}$.

727. IV. As the fluxion of $x^{\frac{m}{n}}$ is $\frac{m}{n}\times x^{\frac{m}{n}-1}\times\dot{x}$, by art. 714 and 719, so the fluxion of a power of any invariable exponent is found by multiplying by the exponent, substracting unit in the index of the power, and multiplying by the fluxion of the root. Thus the respective fluxions of x^2, x^3, x^4, *&c.* are $2x^{2-1}\times\dot{x}$ or $2x\dot{x}$, $3\times x^{3-1}\times\dot{x}$ or $3x^2\dot{x}$, $4\times x^{4-1}\times\dot{x}$ or $4x^3\dot{x}$, &c. In order to give this rule its full extent, and to reduce fluxions to the most simple expressions, we are to suppose from the common algebra, that a quantity may be carried from the numerator of a fraction to its denominator, or from the denominator to the numerator, providing the sign of its index or exponent be changed. Thus the fluxions of $\frac{1}{x}$ or x^{-1}, $\frac{1}{xx}$ or x^{-2}, $\frac{1}{x^3}$ or x^{-3}, are respectively $-1\times x^{-1-1}\times\dot{x}$ or $-\frac{\dot{x}}{x^2}$, $-2\times x^{-2-1}\times\dot{x}$ or $\frac{2\dot{x}}{x^3}$, $-3\times x^{-3-1}\times\dot{x}$ or $-\frac{3\dot{x}}{x^4}$; and the fluxion of $\frac{1}{x^n}$ or x^{-n} is $-n\times x^{-n-1}\times\dot{x}$ or $\frac{-n\dot{x}}{x^{n+1}}$. The fluxions of surds are found by expressing them as powers with fractional exponents. Thus the fluxion of $\sqrt{x}$ or $x^{\frac{1}{2}}$ is $\frac{1}{2}\times x^{\frac{1}{2}-1}\times\dot{x}=\frac{1}{2}\times x^{-\frac{1}{2}}\dot{x}=\frac{\dot{x}}{2x^{\frac{1}{2}}}$ $=\frac{\dot{x}}{2\sqrt{x}}$. The fluxion of $\sqrt[3]{x}$ or $x^{\frac{1}{3}}$ is $\frac{1}{3}\times x^{\frac{1}{3}-1}\times\dot{x}=\frac{x^{-\frac{2}{3}}\dot{x}}{3}$

=

$=\frac{\dot{x}}{3x^{\frac{2}{3}}}=\frac{\dot{x}}{3\sqrt[3]{xx}}$. The fluxion of $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$ is $\frac{1}{n}\times x^{\frac{1}{n}-1}\times\dot{x}$

$=\frac{x^{\frac{1-n}{n}}\dot{x}}{n}=\frac{\dot{x}}{n\,x^{\frac{n-1}{n}}}$; and the fluxion of $\frac{1}{\sqrt[n]{x}}$ or $x^{-\frac{1}{n}}$ is

$-\frac{1}{n}\times x^{-\frac{1}{n}-1}\times\dot{x}=-\frac{x^{\frac{-1-n}{n}}\dot{x}}{n}=\frac{-\dot{x}}{nx^{\frac{1+n}{n}}}$. The flux-ion of $\overline{a+x}^{n}$ is $n\times\overline{a+x}^{n-1}\times\dot{x}$; and the fluxion of $\frac{\overline{a+x}^{m}}{\overline{b+x}^{n}}$

is $\frac{m\times\overline{a+x}^{m-1}\dot{x}\times\overline{b+x}^{n}-n\times\overline{b+x}^{n-1}\times\dot{x}\times\overline{a+x}^{m}}{\overline{b+x}^{2n}}=$

(dividing the numerator and denominator by $\overline{b+x}^{n-1}$)

$\frac{m\dot{x}\times\overline{b+x}\times\overline{a+x}^{m-1}-n\dot{x}\times\overline{a+x}^{m}}{\overline{b+x}^{n+1}}$.

728. V. As when $p=x\times y\times z\times u$, &c. or to this product multiplied by any invariable quantity K, it follows, from art. 719, that $\frac{\dot{p}}{p}=\frac{\dot{x}}{x}+\frac{\dot{y}}{y}+\frac{\dot{z}}{z}+\frac{\dot{u}}{u}$ &c. or that $\dot{p}=\frac{p\dot{x}}{x}+\frac{p\dot{y}}{y}+\frac{p\dot{z}}{z}+\frac{p\dot{u}}{u}$ &c. So the fluxion of any product divided by the product itself is equal to the sum of the quotients, when the fluxion of each factor is divided by the factor; or the fluxion of any product is equal to the sum of the several quantities that are formed, by substituting successively in that product the fluxion of each factor in place of the factor itself. Thus if $p=xyz$,

xyz, then $\dot{p}=\dot{x}yz+\dot{y}xz+\dot{z}xy$. If $p=\overline{a+x}\times\overline{b+x}\times\overline{c+x}$ then $\frac{\dot{p}}{p}=\frac{\dot{x}}{a+x}+\frac{\dot{x}}{b+x}+\frac{\dot{x}}{c+x}$. If $p=\overline{a+x}^{m}$, then $\frac{\dot{p}}{p}=\frac{m\dot{x}}{a+x}$. If $p=\overline{a+x}^{m}\times\overline{b+x}^{n}\times\overline{c+x}^{r}$, then $\frac{\dot{p}}{p}=\frac{m\dot{x}}{a+x}+\frac{n\dot{x}}{b+x}+\frac{r\dot{x}}{c+x}$. If $p=x+\sqrt{xx+1}$, then $\frac{\dot{p}}{p}=\frac{\dot{x}+\frac{x\dot{x}}{\sqrt{xx\mp1}}}{x+\sqrt{xx\mp1}}=\frac{\dot{x}\sqrt{xx\mp1}+x\dot{x}}{\overline{x+\sqrt{xx\mp1}}\times\sqrt{xx\mp1}}=\frac{\dot{x}}{\sqrt{xx\mp1}}$. Hence if $p\mp\sqrt{pp\mp1}=\overline{x\mp\sqrt{xx\mp1}}^{m}$, then $\frac{\dot{p}}{\sqrt{pp\mp1}}=\frac{m\dot{x}}{\sqrt{xx\mp1}}$.

729. VI. As when $p=\frac{x\times y\times z}{s\times u\times t}$ &c. it follows, from art. 717, that $\frac{\dot{p}}{p}=\frac{\dot{x}}{x}+\frac{\dot{y}}{y}+\frac{\dot{z}}{z}-\frac{\dot{s}}{s}-\frac{\dot{u}}{u}-\frac{\dot{t}}{t}$ &c. or $\dot{p}=\frac{p\dot{x}}{x}+\frac{p\dot{y}}{y}+\frac{p\dot{z}}{z}-\frac{p\dot{s}}{s}-\frac{p\dot{u}}{u}-\frac{p\dot{t}}{t}$ &c. So when any fraction is proposed, if we divide the fluxion of each factor of the numerator by the factor itself, and from the sum of the quotients substract the several quotients that arise by dividing the fluxion of each factor of the denominator by this factor itself, the remainder will be equal to the fluxion of the fraction divided by the fraction; or the remainder multiplied by the fraction will give its fluxion. Thus if $p=\frac{a+x}{a-x}\times\frac{b+x}{b-x}\times\frac{c+x}{c-x}$; then $\frac{\dot{p}}{p}=\frac{\dot{x}}{a+x}+\frac{\dot{x}}{a-x}+\frac{\dot{x}}{b+x}+\frac{\dot{x}}{b-x}+\frac{\dot{x}}{c+x}+\frac{\dot{x}}{c-x}=\frac{2a\dot{x}}{aa-xx}+\frac{2b\dot{x}}{bb-xx}+\frac{2c\dot{x}}{cc-xx}$.

730. VII. Any equation of this form $\overline{x-r}\times\overline{x-s}\times\overline{x-u}\times$ &c. $=0$ being proposed, the equation for the fluxions will be $\overline{\dot{x}-\dot{r}}\times\overline{\dot{x}-\dot{s}}\times\overline{\dot{x}-\dot{u}}\times$ &c. $=0$. For since x must be equal to r, or to s, or to u, &c. $\dot{x}$ must be equal to $\dot{r}$, or to $\dot{s}$, or $\dot{u}$, &c.

731. VIII. Let L represent the logarithm of x, the *modulus* being equal to a; then as $\dot{L}=\frac{a\dot{x}}{x}$ by art. 721, so the fluxion

of the logarithm of any quantity is found by dividing its fluxion by the quantity itself, and multiplying by the *modulus*. If $p = x^n$, the fluxion of the logarithm of p is $\frac{a\dot{p}}{p}$ or $\frac{na\dot{x}}{x}$. The fluxion of the logarithm of $x^n y^m$ is $\frac{na\dot{x}}{x} + \frac{ma\dot{y}}{y}$. If $p = \overline{a+x} \times \overline{b+y} \times \overline{c+z} \times$ &c. then the fluxion of the logarithm of p, is $\frac{a\dot{p}}{p}$ or (by art. 728) $\frac{a\dot{x}}{a+x} + \frac{a\dot{y}}{b+y} + \frac{a\dot{z}}{c+z} +$ &c. This likewise follows from the property of logarithms, that the logarithm of the product is equal to the sum of the logarithms of the factors; and consequently the fluxion of the logarithm of p equal to the sum of the fluxions of the logarithms of the factors $a + x$, $b + y$, $c + z$, that is to $\frac{a\dot{x}}{a+x} + \frac{a\dot{y}}{b+y} + \frac{a\dot{z}}{a+x}$. In the same manner the fluxion of the logarithm of $\frac{x-a}{x+a}$ is the difference of the fluxions of the logarithms of $x-a$ and $x+a$, and therefore equal to $\frac{a\dot{x}}{x-a} - \frac{a\dot{x}}{x+a} = \frac{2aa\dot{x}}{xx-xx}$. The fluxion of the logarithm of $\frac{x-a}{x+a} \times \frac{y-b}{y+b} \times$ &c. is $\frac{2aa\dot{x}}{xx-aa} + \frac{2ab\dot{x}}{yy-bb}$ &c.

732. IX. A quantity that has a variable exponent, as y^x, is called an *exponential* or *percurrent** quantity; and its fluxion is $\frac{\dot{x}}{a} \times y^x \times \log. y + xy^{x-1}\dot{y}$. For if we suppose $y^x = u$, then by the properties of logarithms (art. 157) $x \times \log. y = \log. u$. And finding the fluxions by art. 725 and 731, $\dot{x} \times \log. y + \frac{a\dot{y}}{y} \times x = \frac{a\dot{u}}{u}$; consequently the fluxion of y^x, or $\dot{u} = \frac{u\dot{x}}{a} \times \log. y + \frac{yxu}{y} = \frac{\dot{x}}{a} \times y^x \times \log. y + xy^{x-1}\dot{y}$.† In like manner the

* Acta Lips. 1694. † V. Emerson's Fluxions, p. 14. Ex. 18. & 19.

fluxion

fluxions are found of exponential quantities of higher degrees.

733. X. The second fluxion is determined from the first fluxion, and the fluxion of any order from that of the preceding order, by the same rules. It is often useful to suppose one of the variable quantities to flow uniformly, or its fluxion to be constant; in which case that quantity will have no second or higher fluxion, and the second or higher fluxions of quantities that depend upon it will be expressed in a more simple manner. Thus the fluxion of x being supposed constant, the first fluxion of x^n being $n\dot{x}x^{n-1}$, its second fluxion will be $n \times \overline{n-1} \times \dot{x}^2 x^{n-2}$, its third fluxion $n \times \overline{n-1} \times \overline{n-2} \times \dot{x}^3 x^{n-3}$; and its fluxion of any order m will be $n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times$ &c. $\times \dot{x}^m x^{n-m}$, where the factors in the coefficient are to be continued till their number be equal to m.

734. The second or higher fluxions of quantities may be found (without computing those of the preceding orders) by particular theorems, as in the last example. Thus the fluxion of xy is $\dot{x}y + \dot{y}x$; the second fluxion of xy is therefore $\ddot{x}y + 2\dot{x}\dot{y} + x\ddot{y}$; its third fluxion is $\dddot{x}y + 3\ddot{x}\dot{y} + 3\dot{x}\ddot{y} + x\dddot{y}$; and in general the fluxion of xy of any order denoted by m is found by multiplying the fluxion of x of the order m by y, the fluxion of x of the order $m-1$ by $\dot{y}$, the fluxion of x of the order $m-2$ by $\ddot{y}$, and proceeding always in this manner (diminishing the order of the fluxion of x, and increasing the order of the fluxion of y by unit), then prefixing to the several products the respective coefficients of the binomial $1+1$ raised to the power m; the last term being the product of x by the fluxion of y of the order m. If we suppose the fluxion of x to be constant, then the two last terms will give the fluxion of xy of the order required: and if the second fluxion of x be constant, the three last terms will give that fluxion of xy; and so on. When the fluxion of x of any order r, and the fluxion of y of any order s, are supposed constant, the fluxion of xy of any order m (supposing m not to exceed $r+s$) is determined by this theorem.

735. In

735. In the inverse method, it is required to find the fluent when the fluxion is given; and the rules are derived from those of the direct method; as the rules of division and evolution in algebra are deduced from those of multiplication and involution. As when a fluent consists of a variable and an invariable part, the latter does not appear in the fluxion; so when any fluxion is proposed, it is only the variable part of the fluent that can be derived from it. If $\dot{x}$ represent any fluxion that may be proposed, the variable part of the fluent will be equal to x; for supposing y to be any variable quantity, if $x+y$ could represent the fluent of $\dot{x}$, then $\dot{x}+\dot{y}$ would be equal to $\dot{x}$, and $\dot{y}=o$, or y would be invariable, against the supposition. But supposing K to represent any invariable quantity, then $x+\mathrm{K}$ may generally represent the fluent of $\dot{x}$. If it be required to find such a fluent of $\dot{x}$ as shall vanish when x is supposed to vanish, it can be no other than x; and if it be required that the fluent should vanish when x is equal to any given quantity a, then by supposing $x+\mathrm{K}$ to vanish when x becomes equal to a, we have $a+\mathrm{K}=o$, or $\mathrm{K}=-a$; whence the fluent is $x-a$. In the same manner the fluent of $-\dot{x}$ may be generally represented by $\mathrm{K}-x$. When a fluxion, that is proposed, coincides with any of those which were deduced from their fluents in any of the preceding articles, the variable part of the fluent required must coincide with that which was there proposed. As division in algebra leads us to fractions, and evolution to surds, so the inverse method of fluxions leads us often to quantities that are not known in the common algebra, and that cannot be expressed by the common algebraic symbols. In the following articles we will endeavour to give some account of the progress that has been made in this method.

736. I. As the fluxion of $ax+by-cz$ is $a\dot{x}+b\dot{y}-c\dot{z}$; so, conversely, when any aggregate of quantities is proposed, each of which involves a simple fluxion that is not multiplied by any flowing quantity, the variable part of the fluent is found by substituting in place of each fluxion its particular fluent; or by taking away the points, or other fluxionary symbols. Thus the variable part of the fluent of $a\dot{x}+b\dot{y}-c\dot{z}$ is $ax+by-cz$. If it is required that this fluent should vanish when x vanishes, let

y be

y be then equal to e, and z equal to f; and the fluent will be $ax + b \times \overline{y-e} - c \times \overline{z-f}$. For the whole fluent may be expressed by $ax + by - cz + K$, where K is supposed invariable. But, by the supposition, when x vanishes, this fluent vanishes, and is equal to $be - cf + K$; whence $K = -be + cf$; and consequently $ax + by - cz + K$ is equal to $ax + by - be - cz + cf$, or to $ax + b \times \overline{y-e} - c \times \overline{z-f}$.

737. II. As the fluxion of x^n is $nx^{n-1}\dot{x}$, by art. 727, so, conversely, when the fluxion proposed is the product of any power of a variable quantity multiplied by its fluxion, with any invariable coefficient, the variable part of the fluent is found by adding unit to the exponent of the power, dividing by the exponent thus increased and by the fluxion of the root. Thus the variable part of the fluent of $nx^{n-1}\dot{x}$ is $\frac{nx^{n-1+1}\dot{x}}{n-1+1 \times \dot{x}} = x^n$; and if it is required that the fluent should vanish when x vanishes, it is then precisely x^n; but if it is to vanish when x is equal to any given quantity a, the whole fluent is $x^n - a^n$. In general we may express it by $x^n + K$, where K may represent any invariable quantity. In the same manner the fluent of $ax\dot{x}$ is $a \times \frac{x^{1+1}\dot{x}}{2\dot{x}} + K = \frac{1}{2}axx + K$; the fluent of $ax^2\dot{x}$ is $\frac{ax^{2+1}\dot{x}}{3\dot{x}} + K = \frac{1}{3}ax^3 + K$; the fluent of $\frac{a\dot{x}}{x^2}$, or $ax^{-2}\dot{x}$, is $\frac{ax^{-2+1}\dot{x}}{-1 \times \dot{x}} + K = -\frac{a}{x} + K$; the fluent of $x^{\frac{1}{2}}\dot{x}$ is $\frac{2x^{\frac{1}{2}+1}\dot{x}}{3\dot{x}} + K = \frac{2x^{\frac{3}{2}}}{3} + K$. The fluent of an aggregate of quantities of this kind is found by computing the fluent of each term separately. Thus the fluent of $x^2\dot{x} + ax\dot{x} + bb\dot{x}$ is $\frac{1}{3}x^3 + \frac{1}{2}ax^2 + bbx + K$; the fluent of $x\dot{x} \times \overline{a+x}^2$, or of $aax\dot{x} + 2ax^2\dot{x} + x^3\dot{x}$ is $\frac{1}{2}a^2x^2 + \frac{2ax^3}{3} + \frac{x^4}{4}$. The fluent of $x^m\dot{x} \times \overline{x \mp a}^n$ when

when n is an integer, is found by raising $x \mp a$ to the power n, multiplying each term by $x^m \dot{x}$, finding the fluent of each separately by this rule, and collecting them into one sum. The variable part of the fluent is assignable in all those cases, unless when the fluxion $\frac{\dot{x}}{x}$ or $\dot{x}x^{-1}$ is involved in one of the terms, of which case we are to treat afterwards.

738. III. As the fluxion of xy is $\dot{x}y + \dot{y}x$, by art. 725, so when any proposed fluxion can be resolved into two terms of this form, where there are two fluxions, each of which is separately multiplied by the fluent of the other, then the product of the two fluents is the variable part of the fluent required. Thus the fluent of $b\dot{z} - u\dot{z} - a\dot{u} - z\dot{u}$, or $\dot{z} \times \overline{b-u} - \dot{u} \times \overline{a+z}$ is $\overline{b-u} \times \overline{a+z} + K$. In the same manner, when a fluxion can be resolved into three parts in the form $\dot{x}yz + \dot{y}xz + \dot{z}xy$, where there are three fluxions $\dot{x}, \dot{y}, \dot{z}$, and each of these is separately multiplied by the product of the fluents of the other two fluxions, then xyz the product of the three fluents is the variable part of the fluent required. These theorems are easily continued from art. 728.

739. IV. The fluxion of $e\dot{x}z + \dot{z}x$, where e is supposed to be invariable, is not of the same form with any of those in the preceding article, but by multiplying it by x^{e-1}, the product $e\dot{x}x^{e-1}z + \dot{z}x^e$ is easily reduced to the first of them. For supposing $y = x^e$, $e\dot{x}x^{e-1}z + \dot{z}x^e = \dot{y}z + \dot{z}y$, the fluent of which is yz, or zx^e; which is therefore the fluent of $\overline{e\dot{x}z + \dot{z}x} \times x^{e-1}$. In the same manner, if $e\dot{x}yz + f\dot{y}xz + \dot{z}xy$ be multiplied by $x^{e-1}\,y^{f-1}$, the fluent of the product will be $x^e\,y^f\,z$. And when the fluxion $e\dot{x}yzu + f\dot{y}xzu + g\dot{z}xyu + \dot{u}xyz$ is multiplied by $x^{e-1}\,y^{f-1}\,z^{g-1}$, the fluent is $x^e\,y^f\,z^g\,u$; and so on. It follows from the first of these, that when an equation $e\dot{x}z + \dot{z}x = ax^m\dot{x}$ is proposed, the equation for

for the fluents is $z x^{e} + K = \frac{1}{m+e} \times a x^{m+e}$; and in this manner the fluents in art. 540 were found.

740. V. When $\frac{\dot{p}}{p} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} +$ &c. then we may conclude that p is the product of $x, y, z,$ &c. and of some invariable quantity K; for this fluxional equation will arise (by art. 728) when we suppose $p = Kxyz \times$ &c. If $\frac{\dot{p}}{p} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} - \frac{\dot{s}}{s} - \frac{\dot{u}}{u}$ &c. then we may conclude that $p = \frac{Kxyz}{su} \times$ &c. Thus if $\frac{\dot{p}}{p} = \frac{\dot{x}}{x+a} - \frac{\dot{x}}{x+b}$, we may conclude that $p = K \times \frac{x+a}{x+b}$. If $\frac{e\dot{p}}{p} = \frac{m\dot{x}}{x} + \frac{n\dot{y}}{y}$ (e, m and n being supposed invariable), then $p^{e} = K x^{m} y^{n}$.

741. VI. A fluxion that is not proposed under any of the preceding forms may in some cases, by a proper substitution, be changed into an equal fluxion that will appear under one or more of them; and thus the fluent may be discovered. The fluxion $z^{n} \dot{z} \times \overline{a+z}^{m}$ is not immediately comprehended under any of the preceding forms when m is a fraction, or any negative number. But by supposing $x = a + z$, or $z = x - a$, and consequently $\dot{z} = \dot{x}$, and $z^{n} = \overline{x-a}^{n}$ the proposed fluxion is transformed into $\dot{x} x^{m} \times \overline{x-a}^{n}$; the fluent of which is found by raising $x - a$ to the power of the exponent n, multiplying each term by $x^{m} \dot{x}$, and computing the fluent of each product separately by art. 737.

742. The fluxion $x^{m} \dot{x} \times \overline{e+fx^{n}}^{l}$ being proposed; suppose $e + fx^{n} = z$, and $\frac{m+1}{n} = r$; then $x^{n} = \frac{z-e}{f}$, $x^{m+1} = \overline{\frac{z-e}{f}}^{\frac{m+1}{n}} = \overline{\frac{z-e}{f}}^{r}$, and (by taking the fluxions) $\overline{m+1} \times$

$x^{m} \dot{x}$

$\dot{x}^m \dot{x} = \frac{r}{fr} \times \overline{z-e}^{r-1} \times \dot{z}$. Therefore the fluxion that was proposed will be equal to $\frac{r}{\overline{m+1} \times f^r} \times \overline{z-e}^{r-1} \times z^l \dot{z} = \frac{\dot{z} z^l}{nf^r} \times \overline{z-e}^{r-1}$; consequently the fluent is found by raising $z-e$ to the power of the exponent $r-1$, multiplying each term of this power by $\dot{z}z^l$, finding the fluent of each product separately by art. 737, and dividing the sum of these fluents by nf^r. This fluent is assignable in finite terms when r or $\frac{m+1}{n}$ is an integer (unless l be of such a value as to give occasion to the exception mentioned above at the end of art. 737), and will consist of as many terms as there are units in r; because this is the number of terms in the power of $z-e$ of the exponent $r-1$. For example, the fluent of $x^m \dot{x} \times \overline{e+fx}^l$ is assignable in algebraic terms equal in number to $m+1$, when m is any integer and positive number; for in this case $n=1$ and $r=m+1$. The fluent of $x^m \dot{x} \times \overline{e+fx^2}^l$ is assignable in finite terms when m is any odd positive number; because in this case $n=2$, and $r = \frac{m+1}{n} = \frac{m+1}{2}$ which is an integer when m is an odd positive number. The fluxion $x^m \dot{x} \sqrt{ex+fxx} = x^{m+\frac{1}{2}} \dot{x} \times \overline{e+fx}^{\frac{1}{2}}$; and consequently the fluent is assignable when $m+\frac{3}{2}$ is an integer positive number, that is when m is equal to any fraction of this series $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, &c. The fluent of $\dot{x}x^{\frac{1}{s}} \times \overline{e+fx^{\frac{k}{s}}}\Big|^l$ is assignable in finite terms when $s+1$ is any multiple of k; for in this case r (or $\frac{1}{s}+1$ divided by $\frac{k}{s}$) is equal to $\frac{1+s}{k}$, and is an integer when $s+1$ is a multiple of k.

743. The

743. The same fluxion $x^m \dot{x} \times \overline{e+fx^n}|^l$ (multiplying the first part $x^m \dot{x}$ by x^{nl}, and dividing the other part $\overline{e+fx^n}|^l$ by the same x^{nl}, by which their product is not altered) is expressed by $\dot{x}x^{m+nl} \times \overline{ex^{-n}+f}|^l$. As the value of r taken from the first expression was $\frac{m+1}{n}$; so its value computed from the second expression is $\frac{m+nl+1}{-n}$. Therefore the fluent is assignable in a finite number of algebraic terms, not only when $\frac{m+1}{n}$ is an integer and positive number, but likewise when $\frac{m+nl+1}{-n}$ is such a number. Thus the fluent of $\dot{x} \times \overline{e+fx^n}|^{-k-\frac{1}{n}}$ is assignable in a finite number of terms when k is integer and positive, whatever number be represented by n; for in this case $m=0$, $l=-k-\frac{1}{n}=\frac{-nk-1}{n}$, $nl+1=-nk$, and $\frac{m+nl+1}{-n}=\frac{-nk}{-n}=k$.

744. When the difference or sum of two fluents is invariable, their fluxions are equal, as we observed in art. 735. And hence when the same fluxion is represented by two different expressions, as in the two preceding articles, there may be some difference betwixt the fluents that are derived from them by the preceding rules; but by the addition or substraction of an invariable quantity, they will be found to agree with one another. Thus, for example, the fluent of $\dot{x} \times \overline{a+x}|^{-2}$ is (by art. 737) $\frac{\dot{x}}{\dot{x}} \times \overline{a+x}|^{-1} = \frac{1}{a+x}$. The same fluxion is equal to $x^{-2}\dot{x} \times \overline{ax^{-1}+1}|^{-2}$, and the fluent of this fluxion (by the same article) is $\frac{x^{-2}\dot{x} \times \overline{ax^{-1}+1}|^{-1}}{-ax^{-2}\dot{x}} = \frac{\overline{ax^{-1}+1}|^{-1}}{-a} = \frac{-1}{a \times \overline{ax^{-1}+1}}$ $= \frac{-x}{a \times \overline{a+x}}$. The latter fluent vanishes when x vanishes. The former $\frac{1}{a+x}$, by adding the invariable quantity K, becomes

K +

$K + \frac{1}{a+x}$; and if we suppose this fluent to vanish when x vanishes, $K + \frac{1}{a+o} = o$, $K = -\frac{1}{a}$, and the fluent will be $\frac{-1}{a} + \frac{1}{a+x} = \frac{-a-x+a}{a \times \overline{a+x}} = \frac{-x}{a \times \overline{a+x}}$, which coincides with the latter fluent.

745. VII. When a fluent cannot be represented accurately in algebraic terms, it is then to be expressed by a converging series, or by a more simple fluent that is already known. In division in the common algebra (and in decimal arithmetic) the quotient is often such a series. Let $\frac{a\dot{x}}{a-x}$ be the fluxion proposed; and if we divide a by a x by the usual method, we shall find the quotient or $\frac{a}{a-x} = 1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3}$ &c. Hence $\frac{a\dot{x}}{a-x} = \dot{x} + \frac{x\dot{x}}{a} + \frac{x^2\dot{x}}{a^2} + \frac{x^3\dot{x}}{a^3}$ &c. and the fluent of $\frac{a\dot{x}}{a-x}$ is equal (by finding the fluents of the terms $\dot{x}$, $\frac{x\dot{x}}{a}$, $\frac{x^2\dot{x}}{a^2}$, &c. separately from art. 737) to the series $x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \frac{x^4}{4a^3}$ + &c. which may be of use for determining the fluent when x is very small in respect of a; because, in that case, a few terms at the beginning of the series will be nearly equal to the value of the whole. This series gives us the logarithm of $\frac{aa}{a-x}$, the *modulus* being supposed equal to a, by art. 731. For if we suppose $\frac{aa}{a-x} = z$, then $\frac{\dot{x}}{a-x} = \frac{\dot{z}}{z}$, by art. 728, and the fluent of $\frac{a\dot{x}}{a-x}$ is equal to log. z or log. $\frac{aa}{a-x}$, or to — log. $a-x$.

746. In the same manner $\frac{aa}{aa+xx} = 1 - \frac{x^2}{a^2} + \frac{x^4}{a^4} - \frac{x^6}{a^6}$ &c. and the fluent of $\frac{aa\dot{x}}{aa+xx}$ is the fluent of $\dot{x} - \frac{x^2\dot{x}}{a^2} + \frac{x^4\dot{x}}{a^4}$ —

$-\frac{x^6\dot{x}}{7a^6}$ &c. that is (by art. 737), $x-\frac{x^3}{3a^2}+\frac{x^5}{5a^4}-\frac{x^7}{7a^6}$ &c. Because the fluxion of the arch is to the fluxion of its tangent in the duplicate ratio of the radius to the secant (by art. 195), it follows that if the radius be a, the tangent x, and consequently the secant equal to $\sqrt{aa+xx}$, the fluxion of the arch will be equal to $\frac{aa\dot{x}}{aa+xx}$; and the ark itself will be expressed by the series $x-\frac{x^3}{3a^2}+\frac{x^5}{5a^4}-\frac{x^7}{7a^6}$ &c. or $x\times\overline{1-\frac{x^2}{3a^2}+\frac{x^4}{5a^4}-\frac{x^6}{7a^6}\ \&c.}$ This series was given by Mr. *James Gregory* for computing the arch from its tangent. *Commer. epistol.* 1671. Dr. *Halley* has computed the ratio of the circumference of the circle to its diameter from it, by supposing x to be the tangent of an arch of 30 gr. in which case the tangent x is to the secant $\sqrt{aa+xx}$ as 1 to 2, and consequently x to a as 1 to $\sqrt{3}$; so that the arch of 30 gr. is the product of $\frac{a}{\sqrt{3}}$ multiplied by the series $1-\frac{1}{9}+\frac{1}{45}-\frac{1}{189}+\frac{1}{729}$ &c. and the whole circumference to the diameter as $\sqrt{12}$ multiplied by this series to unit. This series may be represented by $1-\frac{1}{3\times3}+\frac{1}{5\times9}-\frac{1}{7\times27}+\frac{1}{9\times81}-\frac{1}{11\times243}$ &c. that the law of its continuation may appear.

747. In like manner, when the roots of powers are extracted by the usual rules in algebra, the root is often expressed by a series of this kind. Thus $\sqrt{aa-xx}=a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}$ &c. consequently $\dot{x}\sqrt{aa-xx}=a\dot{x}-\frac{x^2\dot{x}}{2a}-\frac{x^4\dot{x}}{8a^3}-\frac{x^6\dot{x}}{16a^5}$ &c. Therefore the fluent of $\dot{x}\sqrt{aa-xx}$ is (by art. 737) $ax-\frac{x^3}{6a}-\frac{x^5}{40a^3}-\frac{x^7}{112a^5}-\frac{5x^9}{1152a^7}$ &c. And if CA the radius of the circle be represented by a, upon which CP (*fig.* 298) be taken from the centre C equal to x, CB and PM perpendicular to CA meet the

circle in B and M; then the area CBMP will be expressed by this series; for PM $= aa - xx$, the fluxion of the area CBMP (art. 107) equal to PM $\times \dot{x} = \dot{x}\sqrt{aa-xx}$; and consequently the area CBMP equal to the fluent. Let MN be perpendicular to CB in N, and the area BMN $=$ CBMP — CP $\times$ PM $=$ CBMP $- x\sqrt{aa-xx} = ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5}$ &c. $- ax + \frac{x^3}{2a} + \frac{x^5}{8a^3} + \frac{x^7}{16a^5}$ &c. $= \frac{x^3}{3a} + \frac{x^5}{10a^3} + \frac{3x^7}{56a^5}$ &c.

748. Because the fluxion of the arch BM is to $\dot{x}$ the fluxion of its sine MN or CP, as CB to PM, that is, as a to $\sqrt{aa-xx}$, the fluxion of BM is expressed by $\frac{a\dot{x}}{\sqrt{aa-xx}} = \frac{a\dot{x}\sqrt{aa-xx}}{aa-xx} =$ (dividing the series which expresses $\sqrt{aa-xx}$ by $aa-xx$) $\dot{x} + \frac{x^2\dot{x}}{2a^2} + \frac{3x^4\dot{x}}{8a^4} + \frac{5x^6\dot{x}}{16a^6} +$ &c. consequently the arch BM is equal to $x + \frac{x^3}{6a^2} + \frac{3x^5}{40a^4} + \frac{5a^7}{112a^6}$ &c. $= x + \frac{1\times1}{2\times3} \times \frac{x^2A}{a^2} + \frac{3\times3}{4\times5} \times \frac{x^2B}{a^2} + \frac{5\times5}{6\times7} \times \frac{x^2C}{a^2} +$ &c. where A represents the first term x, B the second term $\frac{x^2A}{6a^2}$, C the third term, and so on. It is useful to represent a series in this manner, that it may be easily continued to any number of terms, and the fluent computed to any degree of exactness that may be required. Let the arch NS described from the centre C meet CM in S, and NS will be to BM as CN to CB, that is as $\sqrt{aa-xx}$ to a; consequently, if the series which expresses BM be multiplied by the series which expresses $\frac{\sqrt{aa-xx}}{a}$, viz. $1 - \frac{x^2}{2a^2} - \frac{x^4}{8a^4} - \frac{x^6}{16a^6}$, &c. the product $x - \frac{x^3}{3a^2} - \frac{2x^5}{15a^4} - \frac{8x^7}{105a^6}$ &c. will represent the ark NS. Therefore MN—NS $= \frac{x^3}{3a^2} + \frac{2x^5}{15a^4} + \frac{8x^7}{105a^6}$ &c. And the area BMN is to CB $\times \overline{\text{MN—NS}}$ as

$\frac{x^3}{3a}$

$\frac{x^3}{3a} + \frac{x^5}{10a^3} + \frac{3x^7}{56a^5}$ &c. to $\frac{x^3}{3a} + \frac{2x^5}{15a^3} + \frac{8x^7}{105a^5}$ &c. or as $1 + \frac{3x^2}{10a^2} + \frac{9x^4}{56a^4}$ &c. to $1 + \frac{2x^2}{5a^2} + \frac{8x^4}{35a^4}$ &c. This ratio by substituting $\frac{c}{b}$ instead of $\frac{x}{a}$ coincides with that which was given in article 655, without a proof, as the ratio (*fig.* 294) of the segment FCO to CD $\times$ $\overline{CF - CS}$.

748. Sir *Isaac Newton*'s binomial theorem is of excellent use for extracting the roots of powers, or reducing a quantity to a series of this kind; and, having made no use of this theorem in demonstrating the rules in the direct method of fluxions, we may the rather give an investigation of it from art. 727. Let it be required to find $\overline{1+x}^n$, where n may represent any integer, number, or fraction, whether it be positive or negative. It is evident, from what is shown in the common algebra concerning powers and their roots, that the first term of any power of $1+x$ is 1, and that the subsequent terms involve x, x^2, x^3, x^4, &c. with invariable coefficients. Suppose, therefore, $\overline{1+x}^n = 1 + Ax + Bx^2 + Cx^3 + Dx^4 +$ &c. where A, B, C, D, &c. represent any such coefficients. By finding the fluxions (art. 727) $n\dot{x} \times \overline{1+x}^{n-1} = A\dot{x} + 2Bx\dot{x} + 3Cx^2\dot{x} + 4Dx^3\dot{x} +$ &c. and, dividing by $n\dot{x}$, we have $\overline{1+x}^{n-1} = \frac{A}{n} + \frac{2Bx}{n} + \frac{3Cx^2}{n} + \frac{4Dx^3}{n} +$ &c. And since this equation must be true, whatever the value of x may be, it follows by supposing $x = o$ (or because the first term of $\overline{1+x}^{n-1}$ must be 1), that $\frac{A}{n} = 1$, and $A = n$. By taking the fluxion of the last equation, $\overline{n-1} \times \overline{1+x}^{n-2} \times \dot{x} = \frac{2B\dot{x}}{n} + \frac{6Cx\dot{x}}{n} + \frac{12Dx^2\dot{x}}{n} +$ &c. and dividing by $\overline{n-1} \times \dot{x}$, we have $\overline{1+x}^{n-2} = \frac{2B}{n \times \overline{n-1}} +$

 $\frac{6Cx}{n \times \overline{n-1}}$

$\frac{6C\dot{x}}{n\times\overline{n-1}} + \frac{12Dx\dot{x}}{n\times\overline{n-1}}$ + &c. and by supposing $x = 0$ (or because the first term of any power of $1 + x$ must be 1), $\frac{2B}{n\times\overline{n-1}} = 1$ or $B = n \times \frac{n-1}{2}$. By taking the fluxions again, we find $\overline{n-1} \times \overline{1+x}^{n-3} \times \dot{x} = \frac{6C\dot{x}}{n\times\overline{n-1}} + \frac{24Dx\dot{x}}{n\times\overline{n-1}}$ &c. and $\overline{1+x}^{n-3} = \frac{6C}{n\times\overline{n-1}\times\overline{n-2}} + \frac{24Dx}{n\times\overline{n-1}\times\overline{n-2}}$ + &c. so that $\frac{6C}{n\times\overline{n-1}\times\overline{n-2}} = 1$, or $C = n \times \frac{n-1}{2} \times \frac{n-2}{3}$; and so on. Therefore $\overline{1+x}^n = 1 + nx + n \times \frac{n-1}{2} \times x^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times x^3$ + &c. And $\overline{a+b}^n = \frac{\overline{a+b}^n}{a^n} \times a^n = \overline{1+\frac{b}{a}}^n \times a^n =$ (by substituting $\frac{b}{a}$ for x) $a^n + \frac{na^nb}{a} + n \times \frac{n-1}{2} \times \frac{a^nb^2}{a^2} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{a^nb^3}{a^3}$ + &c. $= a^n + na^{n-1}b + n \times \frac{n-1}{2} \times a^{n-2}b^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times a^{n-3}b^3$ + &c. which is the binomial theorem.

749. In the same manner if we suppose $\overline{a+bx+cxx+dx^3\ \&c.}^n = A + Bx + Cx^2 + Dx^3$ &c. by supposing $x = 0$, we have $A = a^n$. By taking the fluxions, and dividing by $\dot{x}$, we shall find $\overline{a + bx + cxx\ \&c.}^{n-1} \times \overline{nb + 2ncx + 3ndx^2}$ &c. $= B + 2Cx + 3Dx^2$ + &c. and by supposing $x = 0$, we have $B = na^{n-1}b$. By taking the fluxions again, dividing by $2\dot{x}$, and then supposing $x = 0$, we shall find $C = n \times \frac{n-1}{2} \times a^{n-2}bb + na^{n-1}c$. And by proceeding in the same manner, we may investigate the other coefficients D, E, &c. in Mr. *De Moivre's* theorem for raising a multinomial to any power of the index n.

Of

Of this the reader will find a fuller account in the *Philosoph. Trans. n.* 230, or *Miscel. Analyt. p.* 87. There are several other methods by which these theorems are investigated, but we have described that which is immediately suggested by the method of fluxions, and will be of use afterwards in other enquiries.

750. When any fluxion $\dot{x}$P is proposed, and P is any quantity that can be expressed by any powers of x and invariable quantities, the value of P can be resolved into a series by these theorems; and each term being multiplied by $\dot{x}$, the fluent of each may be found separately by art. 737, such excepted as are of the same form with $A\dot{x}x^{-1}$. Thus to find the fluent of $x^m\dot{x} \times \overline{e + fx^n}^{\,l}$, it is first transformed by supposing $z = e + fx^n$ into $\frac{\dot{z}z^l}{nf^r} \times \overline{z-e}^{\,r-1}$ (as in art. 742) = (by the binomial theorem) $\frac{\dot{z}z^l}{nf^r} \times \overline{z^{r-1} - \overline{r-1} \times z^{r-2}e + \overline{r-1} \times \frac{r-2}{2} \times z^{r-3}ee - \&c.}$ consequently the fluent is the product of $\frac{1}{nf^r}$ multiplied by $\frac{z^{l+r}}{l+r} - \frac{r-1}{l+r-1} \times z^{l+r-1}e + \frac{r-1}{2} \times \frac{r-2}{l+r-2} \times z^{l+r-2}ee$ &c. which (by restoring $e + fx^n$ for z, supposing $l + r = s$, and $rn - n = p$) will be found equal to $\frac{1}{nfs} \times \overline{e + fx^n}^{\,l+1} x^p - \frac{r-1}{s-1} \times \frac{eA}{fx^n} + \frac{r-2}{s-2} \times \frac{eB}{fx^n} - \frac{r-3}{s-3} \times \frac{eC}{fx^n}$ &c. where A represents the first term, B the second, C the third, and so on. This series was given long ago by Sir *Isaac Newton, Commer. epistol.* And in his Treatise of Quadratures he has shown how to assign the fluent of $\dot{x}$P in a series, when P is the product of $a + bx^n + cx^{2n}$ &c. multiplied by x^m, and by any multinomial $e + fx^n + gx^{2n} + hx^{3n}$ &c. raised to a power of any exponent l; or when P is equal to the product of those quanti-

 ties

ties multiplied by $k + l\,x^n + m\,x^{2n}$ &c. raised to a power of any exponent k. *De quadrat. curvar.* prop. 5 & 6.

751. The following theorem is likewise of great use in this doctrine. Suppose that y is any quantity that can be expressed by a series of this form $A + Bz + Cz^2 + Dz^3 +$ &c. where A, B, C, &c. represent invariable coefficients as usual, any of which may be supposed to vanish. When z vanishes, let E be the value of y, and let $\dot{E}$, $\ddot{E}$, $\dddot{E}$, &c. be then the respective values of $\dot{y}, \ddot{y}, \dddot{y}$, &c. z being supposed to flow uniformly. Then $y = E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1 \times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1 \times 2 \times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1 \times 2 \times 3 \times 4\dot{z}^4} +$ &c. the law of the continuation of which series is manifest: for since $y = A + Bz + Cz^2 + Dz^3 +$ &c. it follows that when $z = o$, A is equal to y; but (by the supposition) E is then equal to y; consequently $A = E$. By taking the fluxions, and dividing by $\dot{z}$, $\frac{\dot{y}}{\dot{z}} = B + 2Cz + 3Dz^2 +$ &c. and when $z = o$, B is equal to $\frac{\dot{y}}{\dot{z}}$, that is to $\frac{\dot{E}}{\dot{z}}$. By taking the fluxions again, and dividing by $\dot{z}$ (which is supposed invariable) $\frac{\ddot{y}}{\dot{z}^2} = 2C + 6Dz +$ &c. let $z = o$, and substituting $\ddot{E}$ for $\ddot{y}$, $\frac{\ddot{E}}{\dot{z}^2} = 2C$, or $C = \frac{\ddot{E}}{2\dot{z}^2}$. By taking the fluxions again, and dividing by $\dot{z}$, $\frac{\dddot{y}}{\dot{z}^3} = 6D +$ &c. and by supposing $z = o$, we have $D = \frac{\dddot{E}}{6\dot{z}^3}$. Thus it appears that $y = A + Bz + Cz^2 + Dz^3 +$ &c. $= E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1 \times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1 \times 2 \times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1 \times 2 \times 3 \times 4\dot{z}^4} +$ &c. This proposition may be likewise deduced from the binomial theorem.

Let

Let BD (*fig.* 299), the ordinate of the figure FDM at B, be equal to E, BP $= z$, PM $= y$, and this series will serve for resolving the value of PM, or y (some particular cases being excepted, as when any of the coefficients E, $\frac{\dot{E}}{\dot{z}}$, $\frac{\ddot{E}}{\dot{z}^2}$, &c. become infinite), into a series, not only in such cases as were described in the preceding articles, but likewise when the relation of y and z is determined by an affected equation, and in many cases when their relation is determined by a fluxional equation. This theorem was given by Dr. *Taylor, method. increm.* By supposing the fluxion of z to be represented by BP, or $\dot{z} = z$, we have $y = E + \dot{E} + \frac{\ddot{E}}{2} + \frac{\dddot{E}}{6} + \frac{\ddddot{E}}{24} +$ &c. (as was observed in art. 255); and hence it appears at what rate the fluxion of y of each order contributes to produce the increment or decrement of y, since $y - E = \dot{E} + \frac{\ddot{E}}{2} + \frac{\dddot{E}}{6} + \frac{\ddddot{E}}{24} +$ &c. If Bp be taken on the other side of B equal to BP, then $pm = A - Bz + Cz^2 - Dz^3 +$ &c. $=$ (the same quantities being represented by $\frac{\dot{E}}{\dot{z}}$, $\frac{\ddot{E}}{\dot{z}^2}$, &c. as before, or the base being supposed to flow the same way) $E - \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1\times2\dot{z}^2} - \frac{\dddot{E}z^3}{1\times2\times3\dot{z}^3} + \frac{\ddddot{E}z^4}{1\times2\times3\times4\dot{z}^4} -$ &c. consequently $PM + pm = 2E + \frac{2\ddot{E}z^2}{1\times2\dot{z}^2} + \frac{2\ddddot{E}z^4}{1\times2\times3\times4\dot{z}^4} +$ &c.

752. The area BDMP, or the fluent of $y\dot{z}$, is equal to the fluent of $E\dot{z} + \frac{\dot{E}z\dot{z}}{\dot{z}} + \frac{\ddot{E}z^2\dot{z}}{1\times2\dot{z}^2} + \frac{\dddot{E}z^3\dot{z}}{1\times2\times3\dot{z}^3} +$ &c. that is (because while this area is generated by the ordinate PM, the

quantities $\dot{E}$, $\frac{\ddot{E}}{2}$, $\frac{\dddot{E}}{2\cdot 3}$, &c. are invariable) to $\dot{E}z + \frac{\ddot{E}z^2}{1\times 2\dot{z}} + \frac{\dddot{E}z^3}{1\times 2\times 3\dot{z}^2} + \frac{\ddddot{E}z^4}{1\times 2\times 3\times 4\dot{z}^3} +$ &c. which theorem is not materially different from Mr. *Bernouilli's Act. Erud. Lips.*, 1694. In the same manner the area BDmp $= \dot{E}z - \frac{\ddot{E}z^2}{1\times 2\dot{z}} + \frac{\dddot{E}z^3}{1\times 2\times 3\dot{z}^2} - \frac{\ddddot{E}z^4}{1\times 2\times 3\times 4\dot{z}^3}$ &c. where $\dot{z}$ is supposed to be the same as in the former case; therefore the area PMmp bounded by the ordinates PM and pm, that are at equal distances from BD (or E), on opposite sides, is $2\dot{E}z + \frac{2\dddot{E}z^3}{1\times 2\times 3\dot{z}^2} + \frac{2\dddot{\dot{E}}z^5}{1\times 2\times 3\times 4\times 5\dot{z}^4} +$ &c. and is equal to the rectangle contained by the base Pp (or $2z$) and the series $\dot{E} + \frac{\dddot{E}z^2}{2\times 3\dot{z}^2} + \frac{\dddot{\dot{E}}z^4}{2\times 3\times 4\times 5\dot{z}^4} +$ &c.

753. The series for finding the number of a given logarithm may be deduced by the theorem in art. 751. Let z represent the logarithm of y, the *modulus* being represented by M; and since $\frac{\dot{y}}{y} = \frac{\dot{z}}{M}$ by art. 731, it follows that $\dot{y} = \frac{y\dot{z}}{M}$, $\ddot{y} = \frac{\dot{y}\dot{z}}{M} = \frac{y\dot{z}^2}{M^2}$, $\dddot{y} = \frac{\ddot{y}\dot{z}^2}{M^2} = \frac{y\dot{z}^3}{M^3}$, $\ddddot{y} = \frac{\dddot{y}\dot{z}^3}{M^3} = \frac{y\dot{z}^4}{M^4}$, and so on. When $z = o$, then $y = 1$: therefore we are to suppose $E = 1$, $\dot{E} = \frac{\dot{z}}{M}$, $\ddot{E} = \frac{\dot{z}^2}{M^2}$, $\dddot{E} = \frac{\dot{z}^3}{M^3}$, $\ddddot{E} = \frac{\dot{z}^4}{M^4}$; consequently, by the theorem, $y = 1 + \frac{z}{M} + \frac{z^2}{2M^2} + \frac{z^3}{6M^3} + \frac{z^4}{24M^4} +$ &c. The same series is found by supposing $y = 1 + Az + Bz^2 + Cz^3 + Dz^4 +$ &c.; and therefore $M\dot{y} = y\dot{z} = \dot{z} + Az\dot{z} + Bz^2\dot{z} + Cz^3\dot{z} +$ &c. consequently by finding the fluents, My (or

M

$M + AMz + BMz^2 + CMz^3 + \&c.) = K + z + \frac{Az^2}{2} + \frac{Bz^3}{3} + \frac{Cz^4}{4} + \&c.$ and by comparing the terms of those two values of My that involve the same powers of z, $K = M$, $AM = 1$ or $A = \frac{1}{M}$, $BM = \frac{A}{2}$ or $B = \frac{A}{2M}$, $CM = \frac{B}{3}$ or $C = \frac{B}{3M}$, and so on; therefore $y = 1 + \frac{z}{M} + \frac{z^2}{2M^2} + \frac{z^3}{6M^3} + \&c.$ By supposing $z = M$, $y = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \&c. = 2,7182818$, &c. The ratio of this number to unit is that which Mr. *Cotes* calls the *ratio modularis*, the *modulus* being always the logarithm of this ratio in any logarithmic system. See art. 175.

754. In the same manner the series for finding the cosine, when the arch is given, is deduced from the theorem in art. 751. Let the arch BM $= z$, and its cosine PM $= y$, then because $\dot{y} : \dot{z} :: CP : CM :: \sqrt{a^2 - y^2} : a$, $\frac{\dot{y}^2}{\dot{z}^2} = \frac{a^2 - y^2}{a^2}$, $\frac{2\dot{y}\ddot{y}}{\dot{z}^2} = \frac{-2y\dot{y}}{a^2}$ or $\frac{\ddot{y}}{\dot{z}^2} = \frac{-y}{a^2}$, $\frac{\dddot{y}}{\dot{z}^2} = \frac{-\dot{y}}{a^2}$, $\frac{\ddddot{y}}{\dot{z}^2} = \frac{-\ddot{y}}{a^2} = \frac{y\dot{z}^2}{a^4}$, $\frac{\overset{5}{y}}{\dot{z}^2} = \frac{\dot{y}\dot{z}^2}{a^4}$, and so on. When the arch BM vanishes, or $z = o$, PM $=$ CB $= a$; supposing therefore $E = a$, substitute a for y in those values of $\dot{y}$, $\ddot{y}$, $\dddot{y}$, &c. in order to obtain $\dot{E}$, $\ddot{E}$, $\dddot{E}$, &c. and $\frac{\dot{E}}{\dot{z}} = \frac{a^2 - a^2}{a^2} = o$, $\frac{\ddot{E}}{\dot{z}} = \frac{-a}{a^2} = \frac{-1}{a}$, $\frac{\dddot{E}}{\dot{z}^2} = o$, $\frac{\ddddot{E}}{\dot{z}^4} = \frac{1}{a^3}$, &c. Therefore $y = E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{2\dot{z}^2} + \&c. = a - \frac{z^2}{2a} + \frac{z^4}{24a^3} - \frac{z^6}{720a^5} + \&c.$ If we suppose BM still equal to z, but its right sine MN now equal to y, the same equation $\frac{\dot{y}^2}{\dot{z}^2} = \frac{a^2 - y^2}{a^2}$ will express the relation of $\dot{y}$ to $\dot{z}$, and the values of $\ddot{y}$, $\dddot{y}$, &c. will

will be the same as in the former case: but because when BM vanishes, its sine MN likewise vanishes, we are now to suppose $E = o$, and to substitute o for y in the values of $\dot{y}, \ddot{y}, \dddot{y}$, &c. in order to obtain $\dot{E}, \ddot{E}, \dddot{E}$, &c.; therefore, in this case, $\frac{\dot{E}}{\dot{z}} = \frac{a^2 - o}{a^2} = 1$, $\ddot{E} = o$, $\frac{\dddot{E}}{\dot{z}^3} = \frac{1}{a^2}$, $\ddddot{E} = o$, &c. And $y = z - \frac{z^3}{6a^2} + \frac{z^5}{120a^4} - \frac{z^7}{5040a^6} +$ &c. If y represent the tangent of the ark z, then (as in art. 746) $\frac{\dot{y}}{\dot{z}} = \frac{a^2 + y^2}{a^2}$; and supposing $E = o$ (because y vanishes with z), and, proceeding as before, we shall find $y = z + \frac{z^3}{3a^2} + \frac{2z^5}{15a^4} + \frac{17z^7}{315a^6} +$ &c. If y represent the secant of the ark z, then $\dot{y} : \dot{z} :: y\sqrt{yy - aa} : aa$, and supposing $E = a$, because the secant becomes equal to the radius when the ark vanishes, it will be found that $y = a + \frac{z^2}{2a} + \frac{5z^4}{24a^3} + \frac{61z^6}{720a^5} +$ &c. In the same manner general theorems are found for the reversion of series, such as are given by Sir *Isaac Newton, Commerc. Epist.*, in his letter of *October* 1676, towards the end. We now proceed with our account of the inverse method of fluxions; but will have occasion to return to the doctrine of series afterwards, and to show further the use of the theorems in art. 751 and 752.

CHAP.

CHAP. III.

Of the Analogy betwixt circular Arches and Logarithms, and of reducing Fluents to these, or to hyperbolic and elliptic Arches, or to other Fluents of a more simple Form, when they are not assignable in finite algebraic Terms.

755. WHEN it does not appear that a fluent can be assigned in a finite number of algebraic terms, we are not, therefore, to have recourse immediately to an infinite series. The arches of a circle, and hyperbolic areas or logarithms, cannot be assigned in algebraic terms, but have been computed with great exactness by several methods. By these, with algebraic quantities, any segments of conic sections and the arks of a parabola are easily measured; and when a fluent can be assigned by them, this is considered as the second degree of resolution. When it does not appear that a fluent can be measured by the areas of conic sections, it may however be measured in some cases by their arks; and this may be considered as the third degree of resolution. If it does not appear that a fluent can be assigned by the arks of any conic sections (the circle included), it may however be of some use to assign the fluent by an area or ark of some other figure that is easily constructed or described; and it is often important that the proposed fluxion be reduced to a proper form, in order that the series for the fluent may not be too complex, and that it may not converge at too slow a rate.

756. The rule in art. 737 is of no use to find the fluent of $x^{-1}\dot{x}$, or $\frac{\dot{x}}{x}$; for, according to that rule, the fluent is $\frac{\dot{x} \times x^{-1+1}}{\overline{1-1} \times \dot{x}} = \frac{x^0}{0} =$ (because $x^0 = x^{1-1} = \frac{x}{x} = 1$) $\frac{1}{0}$; from which expression no computation of the fluent can be deduced,

duced, and therefore this case was excepted. By art. 731, the fluent of $\frac{\dot{x}}{x}$ is equal to $\frac{\log. x}{M}$, M being the *modulus*, and the fluent being supposed to vanish when x is equal to 1, or to the quantity whose logarithm vanishes. If we suppose $x = a \mp z$, then $\frac{\mp \dot{a x}}{x} = \frac{a\dot{z}}{a \mp x}$, and the fluent will be found (as in art. 745) $= z \pm \frac{z^2}{2a} + \frac{z^3}{3a^2} \pm \frac{z^4}{4a^3} + \frac{z^5}{5a^4}$ &c. Suppose $p = \frac{a+z}{a-z}$, and (art. 728) $\frac{\dot{p}}{p} = \frac{\dot{z}}{a+z} + \frac{\dot{z}}{a-z}$; consequently the fluent of $\frac{M\dot{p}}{p}$ or $\log. p = 2M \times \overline{\frac{z}{a} + \frac{z^3}{3a^3} + \frac{z^5}{5a^5} + \frac{z^7}{7a^7} + \&c.}$ as in art. 173. In the same manner other theorems are found for computing logarithms.

757. The fluent of $\frac{\dot{x}}{x}$ is equal to AHIE (*fig.*300) the area of the equilateral hyperbola, AH being perpendicular from the vertex A and EI from any point E to the asymptote OH in H and I, supposing OH = 1, and OI = x, O being the centre of the figure; because the ordinate EI $= \frac{AH \times HO}{OI} = \frac{1}{x}$. Hence the area AHIE, or the sector AOE, is called the hyperbolic logarithm of OI, or EI, the *modulus* being supposed equal to AH × OH or 1; and such coincide with the logarithms in *Napier*'s first tables; whereas the tabular logarithms are now equal to these multiplied by the reciprocal of the hyperbolic logarithm of 10, as was more fully explained in art. 174. If the sector OAK : OAE :: n : 1, and KL be perpendicular to the asymptote in L, then log. OL $= n \times$ log. OI; and OL : OH :: OI^n : OH^n.

758. The properties of the circle and ellipse often suggest similar properties of the hyperbola; and reciprocally the properties of hyperbolic areas (which are sometimes more easily discovered because of their analogy to the properties of logarithms described in *book* 1, *chap.* 6) are of use for discovering the analogous properties of circular and elliptic areas. The fol-

following theorem serves to show how great this analogy is, and leads us in a brief manner to various general theorems that relate to the multiplication and division of circular sectors or arks. Let O (*fig.* 300 and 301) be the centre of the ellipse or hyperbola AEK, OA either semi-axis of the ellipse, but the semi-transverse axis in the hyperbola, av the axis perpendicular to OA, OAK a sector that is the same multiple of the sector OAB in both figures, Kk and Bb perpendicular to av in k and b; suppose OA $= a$, Bb $= x$, and K$k = z$, when the perpendiculars Bb, Kk are on the same side of the axis av with OA (as they always are in the hyperbola); but B$b = -x$ or K$k = -z$, when Bb, or Kk, are on the other side of av in the ellipse. Then the relation of z to x will be determined by the same equation in both figures. To make this appear, let AOB, BOC, COD, DOE, &c. be any equal sectors in the hyperbola; and let AOB, BOC, COD, DOE, &c. be likewise any equal sectors in the ellipse; let Bb, Cc, Dd, Ee, &c. be perpendicular to av in b, c, d, e, &c. in each figure; join AC, BD, CE, DF, &c. intersecting the semidiameters OB, OC, OD, OE, &c. in M, N, P, Q, &c. respectively. Because the sectors AOB, BOC, COD, DOE, &c. are equal, AC, BD, CE, DF, &c. are ordinates of the respective semi-diameters OB, OC, OD, OE, &c. For the same reason OB is to OM, OC to ON, OD to OP, OE to OQ, &c. always in the same ratio of Bb to OA, in the same figure; as was shown above of the ellipse (*introd. p.* 8, *&* § 617), and is easily extended to the hyperbola. Let Mm, Nn, Pp, Qq, &c. be perpendicular to the diameter av in each figure in m, n, p, q, r, &c. respectively; and the ratio of Mm to Bb, of Nn to Cc, of Pp to Dd, of Qq to Ee, &c. will be always the same as that of Bb to OA. Then because AC is bisected in M, Cc + OA $= 2$M$m = 2$B$b \times \frac{\text{B}b}{\text{OA}} = 2Bb$ $\times \frac{x}{a}$: because BD is bisected in N, Dd + B$b = 2$N$n = 2$Cc $\times \frac{x}{a}$. In the same manner Ee + C$c = 2$P$p = 2$D$d \times \frac{x}{a}$; and so on: therefore, since in both figures C$c = 2$B$b \times \frac{x}{a} -$OA,

Dd

$Dd = 2Cc \times \frac{x}{a} - Bb$, $Ee = 2Dd \times \frac{x}{a} - Cc$, and so on; it appears that the relation of Cc to Bb, of Dd to Bb, Ee to Bb, and, in general, the relation of Kk to Bb (the sector OAK being the same multiple of OAB in both figures), will be expressed always by the same equation in the ellipse and hyperbola, the perpendiculars Bb and Kk being on the same side of the diameter av with OA. But if the perpendicular Ff (for example) stand in the ellipse on the other side of av, then $-Ff$ will be determined from Bb and OA in the ellipse by an equation of the same form with that which serves for determining $+Ff$ from Bb and OA in the hyperbola; for in this case we find in the ellipse $Dd - Ff = 2Qq = 2Ee \times \frac{x}{a}$, or $-Ff = 2Ee \times \frac{x}{a} - Dd$; and in the hyperbola $+Ff = 2Ee \times \frac{x}{a} - Dd$. In the same manner in the ellipse $-Gg = -2Ff \times \frac{x}{a} - Ee$, but $+Gg = 2Ff \times \frac{x}{a} - Ee$ in the hyperbola; whence $-Ff$, $-Gg$, &c. are determined in the ellipse by the same equation as $+Ff$, $+Gg$, &c. in the hyperbola: and in general it appears that $\mp Kk$ or z is always determined from $\mp Bb$, or x, and OA, or a, in both figures by the same equation.

759. In the equilateral hyperbola, let BS and KT (*fig.* 300) be perpendicular to the transverse axis in S and T, VB and LK perpendicular to the asymptote meet the same axis in X and Z; let the sector OAK : OAB :: n : 1, $OX = y$, Bb or $OS = x$, and Kk or $OT = z$ as before: then by the common property of this hyperbola, $BS^2 = OS^2 - OA^2$, that is $BS = \sqrt{xx-aa}$, and $OX (=y) = OS + SX = OS + BS = x + \sqrt{xx-aa}$: in the same manner $KT = \sqrt{zz-aa}$, $OZ = OT + TZ = OT + TK = z + \sqrt{zz-aa}$. Because the sector AOK : AOB :: n : 1, it follows (art. 757), that $OV^n : OH^n$ (:: $OX^n : OA^n$) :: OL : OH :: OZ : OA; that is, $y^n : a^n :: z + \sqrt{zz-aa} : a$, or

or $z + \sqrt{zz-aa} = \frac{y^n}{a^{n-1}}$, because $y = x + \sqrt{xx-aa}$, $\overline{y-x}^2 = xx - aa$, or $yy - 2xy + aa = o$; and because $z + \sqrt{zz-aa} = \frac{y^n}{a^{n-1}}$, it follows that $y^{2n} - 2a^{n-1}zy^n + a^{2n} = o$. Hence the relation of z to x is found in the hyperbola by comparing the two equations $y^{2n} - 2za^{n-1}y^n + a^{2n} = o$, and $yy - 2xy + aa = o$, and exterminating y. Therefore, by the last art. (*fig.* 301), if the sector OAK be to OAB in the circle as n to 1, or the ark AK $= n \times$ AB, then the relation of $\mp$ Kk (the cosine of the ark AK) to $\mp$ Bb (the cosine of AB) will be determined by supposing $\mp$ K$k = z$, $\mp$ B$b = x$, OA $= a$, and exterminating y from the two equations $y^{2n} - 2za^{n-1}y^n + a^{2n} = o$, and $yy - 2xy + aa = o$; of which theorem Mr. *De Moivre* has made excellent use for resolving a trinomial of the form $y^{2n} - 2zy^n + 1$ into quadratic trinomials (*Miscel. Analyt. lib.* 1), as we shall see afterwards.

760. Produce SB and TK (*fig.* 300), till they meet the asymptote in s and t, and Kt : OA :: Bs^n : OA^n; that is $z - \sqrt{zz-aa} : a :: \overline{x - \sqrt{xx-aa}}^n : a^n$; consequently $z - \sqrt{zz-aa} = a \times \overline{\frac{x - \sqrt{xx-aa}}{a}}\Big|^n$. Therefore since $z + \sqrt{zz-aa} = a \times \overline{\frac{x + \sqrt{xx-aa}}{a}}\Big|^n$, it follows (by adding those equations) that $z = \frac{a}{2} \times \frac{\overline{x + \sqrt{xx-aa}}^n + \overline{x - \sqrt{xx-aa}}^n}{a^n}$, which (by the binomial theorem) is equal to $\frac{1}{a^{n-1}}$ multiplied by $x^n + n \times \frac{n-1}{2} \times x^{n-2} \times \overline{xx-aa} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times x^{n-4} \times \overline{xx-aa}^2$ + &c. Or, the radius a being supposed equal to unit, raise $x+1$ to the power of the exponent n, multiply the terms taken alter-

alternately, beginning with the first x^n by 1, $xx-1$, $\overline{xx-1}|^2$, $\overline{xx-1}|^3$, &c. respectively, and the sum of the products will be equal to z. Hence if Ob the sine of the ark AB (*fig.* 301) be represented by u, or $uu = aa - xx$, then Kk or z will be equal to the product of $\frac{x^n}{a^{n-1}}$ multiplied by $1 - n \times \frac{n-1}{2} \times \frac{uu}{xx} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{u^4}{x^4}$ — &c. It is evident from what has been shown that $x \mp \sqrt{xx-aa} = a \times \overline{\frac{z \mp \sqrt{zz-aa}}{a}}|^{\frac{1}{n}}$ and $2a^{\frac{1+n}{n}}x = \overline{z + \sqrt{zz-aa}}|^{\frac{1}{n}} + \overline{z - \sqrt{zz-aa}}|^{\frac{1}{n}}$. Let O$k$ (the sine of the ark AK) = S, or SS = $aa - zz$ = (by substituting the value of z) $\frac{2a^{2n} - \overline{x + \sqrt{xx-aa}}|^n - \overline{x - \sqrt{xx-aa}}|^n}{4a^{2n-2}}$; consequently $S = \frac{\overline{x + \sqrt{xx-aa}}|^n - \overline{x - \sqrt{xx-aa}}|^n}{2a^{n-1}} \times \sqrt{-1}$ which (by the binomial theorem) is equal to $\frac{1}{a^{n-1}}$ multiplied by $nx^{n-1}u - n \times \frac{n-1}{2} \times \frac{n-2}{3} \times x^{n-3}u^3 +$ &c. The series given by Sir *Isaac Newton* for finding the sine of the ark AK from the sine of AB, may be derived from this theorem, or from article 751.

761. Let Ar and AR (*fig.* 300 and 301), the tangents of the hyperbola or circle at A intercepted by the semidiameters OB and OK, be represented by t and T; and because Bb : Ob : : OA : Ar, we find in the hyperbola $x = \frac{aa}{\sqrt{aa-tt}}$ and $\sqrt{xx-aa} = \frac{at}{\sqrt{aa-tt}}$, but in the circle $x = \frac{aa}{\sqrt{aa+tt}}$ and $\sqrt{xx-aa} = \frac{at\sqrt{-1}}{\sqrt{aa+tt}}$. By substituting these values for x and $\sqrt{xx-aa}$, and similar values for z and $\sqrt{zz-aa}$ in the first equation in art. 759, $z + \sqrt{zz-aa} =$

$a \times$

$a \times \overline{\frac{x+\sqrt{xx-aa}}{a}}\Big|^{n}$, which was shown to be common to both

figures we have in the hyperbola $\frac{a+\mathrm{T}}{\sqrt{aa-\mathrm{TT}}} = \overline{\frac{a+t}{\sqrt{aa-tt}}}\Big|^{n}$, or (because $\frac{\overline{a+\mathrm{T}}^{2}}{aa-\mathrm{TT}} = \frac{a+\mathrm{T}}{a-\mathrm{T}}$) $\frac{a+\mathrm{T}}{a-\mathrm{T}} = \overline{\frac{a+t}{a+t}}\Big|^{n}$, and $\mathrm{T} = a \times \frac{\overline{a+t}^{n} - \overline{a-t}^{n}}{\overline{a+t}^{n} + \overline{a-t}^{n}}$; but in the circle $\frac{a+\mathrm{T}\sqrt{-1}}{\sqrt{aa+\mathrm{TT}}} = \overline{\frac{a+t\sqrt{-1}}{\sqrt{aa+tt}}}\Big|^{n}$

or $\frac{a+\mathrm{T}\sqrt{-1}}{a-\mathrm{T}\sqrt{-1}} = \overline{\frac{a+t\sqrt{-1}}{a-t\sqrt{-1}}}\Big|^{n}$ and $\mathrm{T} = a \times \frac{\overline{a+t\sqrt{-1}}^{n} - \overline{a-t\sqrt{-1}}^{n}}{\overline{a+t\sqrt{-1}}^{n} + \overline{a-\sqrt{t-1}}^{n}}$

$$= (\text{by art. } 748)\ a \times \frac{na^{n-1}t - n \times \frac{n-1}{2} \times \frac{n-2}{3} \times a^{n-3}t^{3} + \&c.}{a^{n} - n \times \frac{n-1}{2} \times a^{n-2}t^{2} + \&c.}$$

This theorem was given by Mr. *Bernouilli, Act. Lips.* 1712.

762. The same theorems are immediately deduced from the inverse method of fluxions, by representing circular arks as imaginary logarithms; for in this manner an analogy is preserved in the expressions of the fluents, as near as possible to that which is betwixt their fluxions, or betwixt the equations of the circle and hyperbola. The fluxion of the hyperbolic sector OAB is to the fluxion of the triangle OAr (or $\frac{1}{2}a\dot{t}$) as BS^{2} to Ar^{2}, or as OS^{2} ($=$ OA^{2} + BS^{2}) to OA^{2}, and consequently as OA^{2} to OA^{2} — Ar^{2}, that is, as aa to $aa-tt$; and is expressed by $\frac{1}{2}a\dot{t} \times \frac{aa}{aa-tt}$; the fluxion of OAK is in the same manner $\frac{1}{2}a\dot{\mathrm{T}} \times \frac{aa}{aa-\mathrm{TT}}$. Therefore since OAK $= n \times$ OAB, we have $\frac{aa\dot{\mathrm{T}}}{aa-\mathrm{TT}} = \frac{naa\dot{t}}{aa-tt}$. By supposing $p = a \times \frac{a+t}{a-t}$, we have (art. 728) $\frac{\dot{p}}{p} = \frac{\dot{t}}{a+t} + \frac{\dot{t}}{a-t} = \frac{2a\dot{t}}{aa-tt}$, and $\frac{aa\dot{t}}{aa-tt} = \frac{a\dot{p}}{2p}$. In the

same manner, by supposing $q = a \times \frac{a+T}{a-T}$, the fluxion $\frac{aa\dot{T}}{aa-TT}$ $= \frac{a\dot{q}}{2q}$; consequently $\frac{a\dot{p}}{2p} = \frac{na\dot{q}}{2q}$, $\frac{\dot{p}}{p} = \frac{n\dot{q}}{q}$, and (art. 738) $p = q^n \times K$ where K is invariable, or $a \times \frac{a+T}{a-T} = a^n K \times \overline{\frac{a+t}{a-t}}\Big|^n$ or (because T and t vanish together, and $a^n K = a$) $\frac{a+T}{a-T} = \overline{\frac{a+t}{a-t}}\Big|^n$, as in the last article.

763. In the same manner the fluxion of the circular ark AB, viz. $\frac{aa\dot{t}}{aa+tt}$ (art. 746), by supposing $p = a \times \frac{a+t\sqrt{-1}}{a-t\sqrt{-1}}$, is transformed into $\frac{a\dot{p}}{2p\sqrt{-1}}$; because (art. 728) $\frac{\dot{p}}{p} = \frac{\dot{t}\sqrt{-1}}{a+t\sqrt{-1}} + \frac{\dot{t}\sqrt{-1}}{a-t\sqrt{-1}} = \frac{2\dot{t}\sqrt{-1}}{aa+tt}$. Therefore the circular ark is equal to the fluent of $\frac{a\dot{p}}{2p\sqrt{-1}}$, and is expressed by $\frac{a}{2M\sqrt{-1}} \times \log. p = \frac{a}{2M\sqrt{-1}} \times \log. a \times \frac{a+t\sqrt{-1}}{a-t\sqrt{-1}}$; where the value of p is imaginary, and is so far compensated by the imaginary symbol $M\sqrt{-1}$, that the whole compound expression may be supposed to denote the circular ark; as such imaginary symbols compensate each other in the expressions of the real roots of cubic and higher equations. See art. 699. In the same manner the fluxion of the circular AK, or $\frac{aa\dot{T}}{aa+TT}$, by supposing $q = a \times \frac{a+T\sqrt{-1}}{a-T\sqrt{-1}}$, is transformed into $\frac{a\dot{q}}{2q}$; and since $\frac{aa\dot{T}}{aa+TT} = \frac{naa\dot{t}}{aa+tt}$, it follows that $\frac{\dot{q}}{q} = \frac{n\dot{p}}{p}$, $q = p^n \times K$, or $a \times \frac{a+T\sqrt{-1}}{a-T\sqrt{-1}} = a^n K \times \overline{\frac{a+t\sqrt{-1}}{a-t\sqrt{-1}}}\Big|^n$, or (because T and t vanish together, and consequently $a^n K = a$) $\frac{a+T\sqrt{-1}}{a+T\sqrt{-1}} = \overline{\frac{a+t\sqrt{-1}}{a-t\sqrt{-1}}}\Big|^n$, as in art. 761.

764. In

.295.N.2.

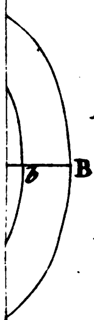


Fig.296.

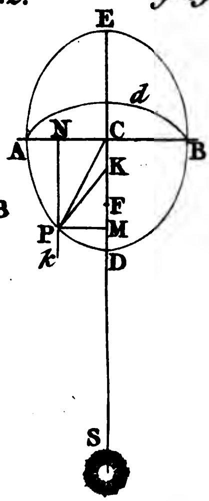
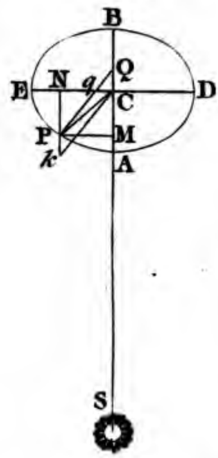


Fig.297.



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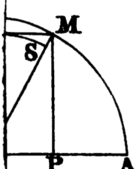


Fig.301.N.2.Art.759.

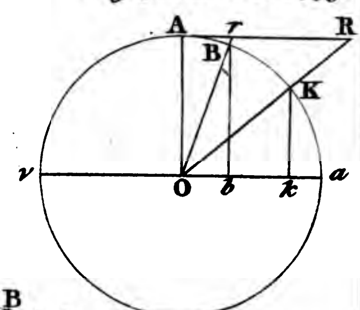


Fig.301.N.1.

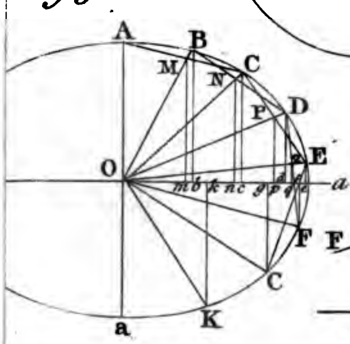
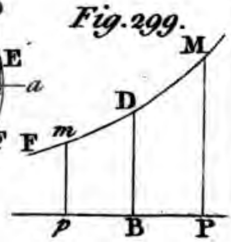


Fig.299.



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764. In like manner, supposing, as above, $OA = a$, $Bb = x$, $Kk = z$, the fluxion of the ark AB (art. 747) is $\frac{-a\dot{x}}{\sqrt{aa-xx}}$, and the fluxion of AK is $\frac{-a\dot{z}}{\sqrt{aa-zz}}$. These fluxions are transformed, by supposing $p = x + \sqrt{xx-aa}$, and $q = z + \sqrt{zz-aa}$ (by art. 728) into $\frac{a\dot{p}}{p\sqrt{-1}}$ and $\frac{a\dot{q}}{q\sqrt{-1}}$. Therefore since AK $= n \times AB$, $\frac{\dot{q}}{q} = \frac{n\dot{p}}{p}$, $q = p^n \times K$, or $z + \sqrt{zz-aa} = \overline{x + \sqrt{xx-aa}}^n \times K =$ (because when Bb or $x = a$, then $z = a$, and consequently $a = a^n K$) $a \times \frac{\overline{x+\sqrt{xx-aa}}^n}{a^n}$, as we found in art. 759. And supposing $y = x + \sqrt{xx-aa}$, the relation of z to x will be determined by exterminating y from the equations $y^{2n} - 2za^{n-1}y^n + a^{2n} + o$, and $y^2 - 2zy + a^2 = o$, as we demonstrated in art. 759, without making use of the imaginary sign, by showing that the relation of z to x must be determined by the same equations in the circle and hyperbola.

765. Let (*fig.* 302) the circumference of the circle be represented by C, the radius OA by 1, the ark AK by A, and consequently AB by $\frac{A}{n}$. Let the circumference be divided into as many equal parts (beginning from the point B) BC, CD, DE, EF, FG, GB, as there are units in n; then $AB = \frac{A}{n}$, $AG = BG - AB = \frac{C-A}{n}$, $AC = AB + BC = \frac{C+A}{n}$, $AF = \frac{2C-A}{n}$, $AD = \frac{2C+A}{n}$, and $AE = \frac{3C-A}{n}$. The cosine Kk of the ark AK being represented by $\mp z$, as before, let the cosines of those several arks AB, AG, AC, AF, AD, AE, &c. be represented by $\mp a$, $\mp b$, $\mp c$, $\mp d$, $\mp e$, $\mp f$, &c. respectively, where the sign of each cosine is supposed positive or negative accord-

 ing

ing as it is on the same side of the diameter au, with OA, or not. Then because those arks are in the same ratio of 1 to n to the several arks A, C—A, C+A, 2C—A, 2C+A, 3C—A, 3C+A, &c. which have all the same cosine $Kk = \mp z$; the several relations of z to a, z to b, z to c, z to d, &c. are found by comparing successively the equation $y^{2n} - 2zy^n + 1 = o$, with the several quadratic equations $yy - 2ay + 1 = o$, $yy - 2by + 1 = o$, $yy - 2cy + 1 = o$, $yy - 2dy + 1 = o$, &c. and always exterminating y. From which it follows, that the trinomials $yy - 2ay + 1$, $yy - 2by + 1$, $yy - 2cy + 1$, $yy - 2dy + 1$, &c. are divisors of $y^{2n} - 2zy^n + 1$, or this last is equal to the product of those trinomials.

766. Upon OA take OP to represent y, join PB, PC, PD, PE, &c. and $PB^2 = Ob^2 + \overline{Bb \pm OP}^2 = Ob^2 + Bb^2 \pm 2Bb \times OP + OP^2 = 1 - 2ay + yy$. In the same manner $PG^2 = 1 - 2by + yy$, $PC^2 = 1 - 2cy + yy$, &c.; consequently $PB^2 \times PG^2 \times PC^2 \times PF^2 \times PD^2 \times$ &c. $= y^{2n} - 2zy^n + 1 = y^{2n} \pm 2Kk \times y^n + 1$. Let the ark AK or A be now supposed equal to the whole circumference C, then (*fig.* 303) $BA = \frac{A}{n} = \frac{C}{n} = BG$, and G coincides with A. In this case the point K falls on the point A, $+ z = Kk = OA = 1$; and $PB^2 \times PA^2 \times PC^2 \times PD^2 \times$ &c. $= y^{2n} - 2y^n + 1$; and $PB \times PA \times PC \times PD \times$ &c. $= y^n - 1$ or $1 - y^n$. From which it follows, that if the circumference be divided into as many equal parts at A, B, C, D, &c. as there are units in n, upon OA $(= 1)$ you take $OP = y$, and from P draw right lines to the other points B, C, D, E, F, &c.; then the product of all those right lines, $PA \times PB \times PC \times PD \times$ &c. will be equal to $y^n - 1$, or $1 - y^n$, that is, to $OP^n - OA^n$, or $OA^n - OP^n$, according as OP is greater or less than OA; which coincides with the first part of the elegant theorem invented by Mr. *Cotes*, and described by Dr. *Smith*, *Harmon. Mensurar. p.* 114. Let the semidiameter AO produced meet the circle in a, and if n be an even number, one of the points wherein the circumference is supposed to be divided will fall

on

on a; consequently the divisors of $1 - y^n$ will be the rectangle APa with the squares of the right lines PB, PC, &c. that are on one side of Aa; but if n be an odd number, PA (or $1 - y$) will be a simple divisor of $1 - y^n$, and the same squares of PB, PC, &c. will be its other divisors. Suppose now the ark AK equal to the semi-circumference, or to $\frac{1}{2}$C (*fig.* 303); then BA $= \frac{A}{n} = \frac{C}{2n} = \frac{1}{2}$BG; and in this case K falls upon a, K*k*, or Oa, $= -z = 1$, or $z = -1$. From which it appears, that if the circumference be divided at B, C, D, E, &c. into as many equal parts as there are units in n, and one of those parts BG being bisected in A, you join OA, and take OP upon it to represent y, then from P draw the right lines PB, PC, PD, PE, &c. to the several divisions of the circumference; the product of the squares of all those lines $PB^2 \times PC^2 \times PD^2 \times PE^2 \times$ &c. will be equal to $y^{2n} \times 2zy^n + 1$; and consequently $PB \times PC \times PD \times PE \times$ &c. $= 1 + y^n = OA^n + OP^n$; which coincides with the latter part of Mr. *Cotes*'s theorem. When n is an even number, the same product is that of the squares of the several right lines PB, PC, &c. that are on the same side of the diameter A*a*; which are therefore the quadratic divisors of $1 + y^n$ in this case; but when n is an odd number, one of the divisions of the circumference falls upon a; and the same squares with Pa, or $1 + y$, are the divisors of $1 + y^n$. By supposing AK (*fig.* 304) equal to a quadrant of the circle, or to $\frac{1}{4}$ C, or $z = o$, it will appear that the circumference being divided as before, if the ark BA be taken upon BG equal to $\frac{1}{4}$ BG, and upon OA you take OP $= y$, then $PB^2 \times PC^2 \times PD^2 \times$ &c. $= 1 + y^{2n} = OA^{2n} + OP^{2n}$. The reader will find this subject treated in a different manner, *Epist. ad amicum de Cotesii inventis*, 1722.

767. In general, the circumference being divided into the same number of equal parts, let P (*fig.* 300) be any point in the plane of the circle, let OP meet the circumference in A; take the ark AK $= n \times$ AB, and upon OA take OQ :

$OP :: OP^{n-1} : OA^{n-1}$, join QK; and $PB^2 \times PC^2 \times PD^2 \times PE^2 \times$ &c. $= QK^2 \times OA^{2n-2}$; because Kk being supposed equal to $+ z$, or $- z$ (according as Kk is on the same side of av with OA, or on a different side), then $QK^2 = 1 - 2z \times OQ + OQ^2 = 1 - 2zy^n + y^{2n}$. In this manner Mr. *Cotes's* theorem was rendered more general by Mr. *De Moivre*. Hence several other propositions relating to the circle may be briefly derived. By supposing OP to concide with OA, it appears that the product of the chords $AB \times AC \times AD \times AE \times$ &c. $= AK \times OA^{n-1}$. For in this case OA, OP, and OQ, being equal, $AB^2 \times AC^2 \times AD^2 \times AE^2 \times$ &c. $= AK^2 \times OA^{2n-2}$, and $AB \times AC \times AD \times$ &c. $= AK \times OA^{n-1}$; which is demonstrated in a different manner, *Hospital. sect. coniq. lib.* 10, *theor.* 1 *and* 3.

768. The quadratic divisors of a trinomial may be likewise discovered from the common algebra. To resolve a quantity as $y^{2n} - 2zy^n + 1$ into its divisors, is a problem equivalent to the resolution of the equation $y^{2n} - 2zy^n + 1 = o$. By proceeding as is usual in the resolution of quadratic equations, $y^{2n} - 2zy^n + zz = zz - 1$, $y^n - z = \mp \sqrt{zz-1}$; and the divisors are $y^n - z + \sqrt{zz-1}$, and $y^n - z - \sqrt{zz-1}$, the product of which is $y^{2n} - 2zy^n + 1$. When z is less than 1, $\sqrt{zz-1}$ is imaginary, and those divisors involve imaginary expressions. But we are not thence to conclude that other divisors cannot be assigned in this case, which may involve real quantities only. It is obvious that $\overline{y-a} \times \overline{y-b} \times \overline{y-c} \times \overline{y-d}$ may be resolved into several different pairs of quadratic divisors, as into $\overline{y-a} \times \overline{y-b}$, and $\overline{y-c} \times \overline{y-d}$, or into $\overline{y-a} \times \overline{y-c}$, and $\overline{y-b} \times \overline{y-d}$; and though the first two may involve the imaginary symbol, the latter may involve no quantities but such as are real. Thus supposing $y^4 - 2zy^2 + 1 = o$, we have $y^2 = z + \sqrt{zz-1}$ or $z - \sqrt{zz-1}$; and the four simple divisors (by extracting the square root again) are in this case $y + \sqrt{z + \sqrt{zz-1}}$, $y - \sqrt{z + \sqrt{zz-1}}$,

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$y + \sqrt{z - \sqrt{zz-1}}$, and $y - \sqrt{z - \sqrt{zz-1}}$. The product of the first and second gives $yy - z - \sqrt{zz-1}$; and the product of the third and fourth gives $yy - z + \sqrt{zz-1}$, the same intermediate divisors from which the simple divisors were derived; both of which involve an imaginary quantity when z is less than 1. But the product of the first and third gives $yy + \overline{\sqrt{z + \sqrt{zz-1}} + \sqrt{z - \sqrt{zz-1}}} \times y + 1 =$ (because the square of $\sqrt{z + \sqrt{zz-1}} + \sqrt{z - \sqrt{zz-1}}$ is $z + \sqrt{zz-1} \mp 2 + z - \sqrt{zz-1} = 2z \mp 2$) $yy + \sqrt{2z \mp 2} \times y + 1$, or $yy - \sqrt{z2 \mp 2} \times y + 1$; and the product of the third and fourth agrees with these. Thus we find $y^4 - 2zy^2 + 1 = yy + \sqrt{2z \mp 2} \times y + 1 \times yy - \sqrt{2z \mp 2} \times y + 1$. And these divisors may involve no imaginary quantity, though z be supposed negative, and less than unit. By continuing the resolution till we have the simple divisors, and then compounding those divisors together variously, quadratic divisors may be formed in this manner, some of which will have all their coefficients real when z is greater than unit, and others when z is less than unit; and we are not to conclude that no real quadratic divisors can be assigned, because those of one combination are imaginary. The latter quadratic divisors are likewise found by resolving the equation $y^4 - 2zy^2 + 1 = o$ in a manner somewhat different from the usual way of proceeding; for since $y^4 + 1 = 2zy^2$, complete the square on the first side of the equation by adding the middle term $\mp 2y^2$, and $y^4 \mp 2y^2 + 1 = + 2zy^2 \mp 2y^2$; consequently by extracting the square root, $y^2 \mp 1 = \mp \sqrt{2z \mp 2} \times y$; and the quadratic divisors are $y^2 - \sqrt{2z \mp 2} \times y + 1$, and $y^2 + \sqrt{2z \mp 2} \times y + 1$, as before. By the same method the divisors of $y^{2n} - 2zy^n + 1$ are $y^n \mp \sqrt{2z \mp 2} \times y^{\frac{n}{2}} + 1$, which may be again resolved in the same manner into divisors of inferior dimensions; and by a continual bisection of the exponent, when n is any power of the number 2, we may at length find the quadratic divisors. But this last method is not applicable when n is any other number.

769. Supposing therefore $y^{2n} - 2zy^n + 1 = o$, as before; and consequently $y^n = z + \sqrt{zz-1}$ or $z - \sqrt{zz-1}$; then by

evolution $y = \sqrt[n]{z + \sqrt{zz-1}}$ or $\sqrt[n]{z - \sqrt{zz-1}}$; consequently two of the simple divisors are $y - \sqrt[n]{z+\sqrt{zz-1}}$ and $y - \sqrt[n]{z-\sqrt{zz-1}}$; and, the quadratic divisor arising from their multiplication by each other being $yy - \overline{\sqrt[n]{z+\sqrt{zz-1}} + \sqrt[n]{z-\sqrt{zz-1}}} \times y + 1$, suppose the coefficient of the middle term to be $2x$, or this quadratic divisor to be $yy - 2xy + 1$, and $2x = \sqrt[n]{z+\sqrt{zz-1}} + \sqrt[n]{z-\sqrt{zz-1}}$. By comparing this equation with that which was deduced in art. 759, it will appear that, when z is less than 1, if z be the cosine of a circular ark A, then x will be the cosine of an ark equal to $\frac{A}{n}$. Or if we suppose $z + \sqrt{zz-1} = p$ and $z - \sqrt{zz-1} = q$, and consequently $2z = p + q$, $x = \frac{p^{\frac{1}{n}} + q^{\frac{1}{n}}}{2}$, $\sqrt{xx-1} =$ (because $pq = 1$) $\frac{p^{\frac{1}{n}} - q^{\frac{1}{n}}}{2}$, $x + \sqrt{xx-1} = p^{\frac{1}{n}}$ and $x - \sqrt{xx-1} = q^{\frac{1}{n}}$; so that $2z = \overline{x+\sqrt{xx-1}}^{n} + \overline{x-\sqrt{xx-1}}^{n}$, the same equation that we found in art. 759, for the cosines; which, expanded as above, will be found always to agree with those by which the relations of the cosines are determined by the common methods. But let us now proceed to show how fluents, or areas, are measured by circular arks and logarithms; and, first, when the ordinates are expressed by rational quantities.

770. Let it be required to assign the fluent of $\frac{y^n \dot{y}}{y-a}$, n being any integer positive number. It was shown in art. 709, that if $y^{n-1} + y^{n-2}a + y^{n-3}a^2 \ldots a^{n-1}$ be multiplied by $y - a$, the product will be $y^n - a^n$. Therefore $\frac{y^n \dot{y}}{y-a} = y^{n-1}\dot{y} +$

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$ay^{n-2}\dot{y} + a^2y^{n-3}\dot{y} \ldots + a^{n-1}\dot{y} + \frac{a^n\dot{y}}{y-a}$; consequently the fluent of $\frac{y^n\dot{y}}{y-a}$ (by art. 737 and 740) is $\frac{y^n}{n} + \frac{ay^{n-1}}{n-1} + \frac{a^2y^{n-2}}{n-2} \ldots + a^{n-1}y + \frac{a^n}{M} \times \log.\overline{y-a}$. In the same manner $y^{n-1} - y^{n-2}a + y^{n-3}a^2 \ldots \mp a^{n-1} = \frac{y^n \pm a^n}{y+a}$. Therefore the fluent of $\frac{y^n\dot{y}}{y+a}$ is $\frac{y^n}{n} - \frac{ay^{n-1}}{n-1} + \frac{a^2y^{n-2}}{n-2} \ldots \mp \frac{a^n}{M} \times \log.\overline{y+a}$.

771. Any integer number being represented by n, the fluent of $\frac{y^n\dot{y}}{aa+yy}$ is expressed by a circular ark, or logarithm (with algebraic quantities), according as n is an even or odd number. For it appears, as in the last article, that when n is an even positive number, if $y^{n-2} - a^2y^{n-4} + a^4y^{n-6} - a^6y^{n-8}$ + &c. be multiplied by $y^2 + a^2$, the product will be $y^n - a^n$, or $y^n + a^n$, according as $\frac{1}{2}n$ is an even or odd number. Therefore $\frac{y^n\dot{y}}{yy+aa} = y^{n-2}\dot{y} - a^2y^{n-4}\dot{y} + a^4y^{n-6}\dot{y} \ldots \mp \frac{a^n\dot{y}}{yy+aa}$; consequently if A represent the ark whose tangent is equal to y, the radius being equal to a (so that $\dot{A} = \frac{aa\dot{y}}{yy+aa}$, by art. 744), the fluent of $\frac{y^n\dot{y}}{yy+aa}$ will be equal to $\frac{y^{n-1}}{n-1} - \frac{a^2y^{n-3}}{n-3} + \frac{a^4y^{n-5}}{n-5} \ldots \mp a^{n-2} \times A$. When n is an odd affirmative number, suppose it equal to $m + 1$; and, by what has

has been shown, $\frac{y^{m+1}\dot{y}}{yy+aa} = y^{m-1}\dot{y} - y^{m-3}a^2\dot{y} + y^{m-5}a^4\dot{y}\ldots$ $\mp \frac{a^m y\dot{y}}{yy+aa}$; the fluent of which (because the fluent of $\frac{y\dot{y}}{aa+yy}$ is $\frac{\log.\sqrt{aa+yy}}{M}$) is $\frac{y^m}{m} - \frac{a^2y^{m-2}}{m-2}\ldots + \frac{a^m}{M} \times \log.\sqrt{aa+yy}$ $= \frac{y^{n-1}}{n-1} - \frac{a^2y^{n-3}}{n-3} + \frac{a^4y^{n-5}}{n-5}\ldots \mp \frac{a^{n-1}}{M} \times \log.\sqrt{aa+yy}$.

By supposing $y = \frac{aa}{z}$, $\frac{y^{-n}\dot{y}}{aa+yy}$ is transformed into $-\frac{z^n\dot{z}}{aa+zz} \times \frac{1}{a^{2n}}$, the fluent of which (by what has been shown) is expressed by a circular ark or logarithm, according as n is an even or odd number. By supposing $z = a \times \frac{y-a}{y+a}$, the fluxion $\frac{aa\dot{y}}{yy-aa}$ is transformed into $\frac{a\dot{z}}{2z}$, by art. 728; consequently the fluent is $\frac{a}{2M} \times \log. a \times \frac{y-a}{y+a}$; and the fluent of $\frac{y\dot{y}}{yy-aa}$ being equal to $\frac{\log.\sqrt{yy-aa}}{M}$, it easily follows that when n is any integer number, the fluent of $\frac{y^n\dot{y}}{yy-aa}$ is expressed by logarithms and algebraic quantities.

772. Let $\frac{\dot{y}}{aa+2by+yy} = \dot{Q}$, and let it be required to find the fluent of $\dot{Q}$. By supposing $y + b = z$, and consequently $\dot{y}=\dot{z}$, $yy + 2by + aa = zz + aa - bb$, $\dot{Q} = \frac{\dot{z}}{zz+aa-bb}$, and the fluent Q is expressed by a circular ark, or logarithm, according as a is greater or less than b, by the last article; and if $a = b$ the fluent is $-\frac{1}{z} + K$, by art. 737. The fluxion $\frac{y\dot{y}}{aa+2by+yy}$ is transformed into $\frac{z\dot{z}-b\dot{z}}{zz+aa-bb}$, by the same substitution;

tion; and the fluent may be found by the last article. Or supposing $\sqrt{aa+2by+yy} = u$, because $\frac{y\dot{y}+b\dot{y}}{aa+2by+yy}$ is equal to $\frac{\dot{u}}{u}$, by art. 728, it follows that the fluent of $\frac{y\dot{y}}{aa+2by+yy}$ is $\frac{\log. u}{M} - bQ = \frac{\log. \sqrt{aa+2by+yy}}{M} - bQ$. The fluent of $\frac{y^n\dot{y}}{aa+2by+yy}$ is found (when n is any integer positive number) by dividing y^n by $yy + 2by + aa$, and continuing the operation till the remainder be of the form $Ay + B$ (where A and B represent invariable coefficients), multiplying each term of the quotient by $\dot{y}$, finding the fluent of each product by art. 737, and determining the fluent of $\frac{Ay\dot{y} + B\dot{y}}{aa+2by+yy}$, by this article. The fluxion $\frac{y^{-n}\dot{y}}{1+2by+yy}$ is transformed, by supposing $y = \frac{1}{z}$, into $\frac{-z^n\dot{z}}{1+2bz+zz}$, and the fluent is found as before. It appears, therefore, that, the ordinate being expressed by a fraction, if the denominator be any quadratic trinomial $1 + 2by + yy$, and the numerator consist of terms that involve any powers of y and invariable quantities; and the exponents of those powers of y be integers; the fluent may be assigned by circular arks or logarithms with algebraic quantities. And any fluxion $P\dot{y}$ being proposed, if P can be resolved into any number of fractions of this form, the fluent of $P\dot{y}$ can be assigned in like manner.

773. What was demonstrated in art. 715 and 717, or in 728 and 729, is often of use for resolving an ordinate into such fractions. For example, as when $p = xyzu \times$ &c. it follows, that $\frac{\dot{p}}{p} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} +$ &c. so if we resolve $1 + y^n$ (supposing n to be an even number) into its quadratic divisors $1 - 2ay + yy$, $1 - 2by + yy$, $1 - 2cy + yy$, &c. according to art. 765, it follows, that $\frac{ny^{n-1}\dot{y}}{1 + y^n} = \frac{2y\dot{y} - 2a\dot{y}}{1-2ay+yy} + \frac{2y\dot{y} - 2b\dot{y}}{1-2by+yy}$

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$+\frac{2y\dot{y}-2c\dot{y}}{1-2cy+yy}$ + &c. and consequently $\frac{ny^{n}\dot{y}}{1+y^{n}} = \frac{2yy\dot{y}-2ay\dot{y}}{1-2ay+yy} + \frac{2yy\dot{y}-2by\dot{y}}{1-2by+yy}$ + &c. $=$ (because $\frac{2yy-2ay}{1-2ay+yy} = \frac{2yy-4ay+2+2ay-2}{1-2ay+yy} = 2+\frac{2ay-2}{1-2ay+yy}$) $n\dot{y} + \frac{2ay-2}{1-2ay+yy} + \frac{2by-2}{1-2by+yy} + \frac{2cy-2}{1-2cy+yy}$ + &c. so that the fluent of $\frac{y^{n}\dot{y}}{1+y^{n}}$ is equal to y added to the fluents of the several fluxions $\frac{2ay-2}{1-2ay+yy}\times\frac{\dot{y}}{n}$, $\frac{2by-2}{1-2by+yy}\times\frac{\dot{y}}{n}$, &c. which are found by the last article.

774. The same method serves for investigating briefly the first four propositions of Mr. *De Moivre's Miscel. Analyt. lib.* 1. Suppose, first, n to be an even number, and since $1+y^{n} = \overline{1-2ay+yy}\times\overline{1-2by+yy}\times\overline{1-2cy+yy}\times$ &c. it follows, dividing $1+y^{n}$ by y^{n}, and each quadratic divisor by yy, that $\frac{1+y^{n}}{y^{n}} = \overline{1-2ay^{-1}+y^{-2}}\times\overline{1-2by^{-1}+y^{-2}}\times$ &c. Therefore, as when $\frac{p}{q} = xyzu\times$ &c. $\frac{\dot{p}}{p}-\frac{\dot{q}}{q} = \frac{\dot{x}}{x}+\frac{\dot{y}}{y}+\frac{\dot{z}}{z}$ + &c. (by art. 728), so in this case $\frac{ny^{n-1}\dot{y}}{1+y^{n}} - \frac{n\dot{y}}{y}$ ($= \frac{-n\dot{y}}{1+y^{n}}\times\frac{1}{y}$) $= \frac{2ay^{-2}\dot{y}-2y^{-3}\dot{y}}{1-2ay^{-1}+y^{-2}} + \frac{2by^{-2}\dot{y}-2y^{-3}\dot{y}}{1-2by^{-1}+y^{-2}}$ + &c. or (dividing both sides by $-\dot{y}y^{-1}$) $\frac{n}{1+y^{n}} = \frac{2-2ay}{1-2ay+yy} + \frac{2-2by}{1-2by+yy} + \frac{2-2cy}{1-2cy+yy}$; which is the first of those propositions. When n is an odd number, then, besides the quadratic divisors of $1+y^{n}$ (which, according to art. 766 (*fig.* 304), are PB^2, PC^2, PD^2

PD^2, &c.), there is a simple divisor $Pa = 1 + y$; and it appears that in this case $\frac{n}{1+y^n} = \frac{1}{1+y} + \frac{2-2ay}{1-2ay+yy} + \frac{2-2by}{1-2by+yy}$ + &c. When n is an even number, the rectangle APa, or $1-yy$, is one of the divisors of $1-y^n$ (by art. 766), and the other quadratic divisors PB^2, PC^2, PD^2, &c. being expressed by $1-2ay+yy$, $1-2by+yy$, &c. as before, it appears, in the same manner, that in this case $\frac{n}{1-y^n} = \frac{2}{1-yy} + \frac{2-2ay}{1-2ay+yy} + \frac{2-2by}{1-2by+yy}$ + &c.; and when n is an odd number, PA or $1-y$ being one of the divisors of $1-y^n$ (by art. 766), we shall find in the same manner that $\frac{n}{1-y^n} = \frac{1}{1-y} + \frac{2-2ay}{1-2ay+yy} + \frac{2-2by}{1-2by+yy} +$ &c. The ordinate $\frac{1}{1 \mp y^n}$ being resolved in this manner into fractions with quadratic denominators, the area or the fluent of $\frac{\dot{y}}{1 \mp y^n}$ (and consequently of $\frac{y^m \dot{y}}{1 \mp y^n}$ when m is any integer number) is reduced to circular arks or logarithms with algebraic quantities, by art. 772. The ordinate $\frac{1}{e+fy^n+gy^{2n}}$ is resolved into fractions of the same kind by this method, when ff is greater than $4eg$; that is, when the roots of the quadratic equation $y^{2n} + \frac{fy^n}{g} + \frac{e}{g} = o$ are real. For supposing those roots to be $-R^n$ and $-r^n$; or $e + fy^n + gy^{2n} = g \times \overline{y^n + R^n} \times \overline{y^n + r^n}$, let $y^n + R^n$ and $y^n + r^n$ be resolved into their respective divisors (art. 766) $RR - 2Ay + yy$, &c. and $rr - 2ay + yy$, &c.; and, by what has been shown, $\frac{n}{r^n + y^n} - \frac{n}{R^n + y^n}$ or

or $\frac{R^n - r^n}{e + fy^n + gy^{2n}} \times ng = \frac{1}{r^n} \times \frac{2rr - 2ay}{rr - 2ay + yy} + \&c. - \frac{1}{R^n} \times \frac{2RR - 2Ay}{RR - 2Ay + yy} - \&c.$

775. When z is less than 1, let $y^{2n} - 2zy^n + 1$ be resolved (by art. 766) into its divisors $1 - 2ay + yy$, $1 - 2by + yy$, &c. Then (z being supposed invariable) it follows, as in art. 728, that $\frac{ny^{2n-1} - nzy^{n-1}}{y^{2n} - 2zy^n + 1} = \frac{y-a}{1-2ay+yy} + \frac{y-b}{1-2by+yy} + \&c.$ or (multiplying by $\frac{z}{y^{n-1}}$) $\frac{nzy^n - nzz}{y^{2n} - 2zy^n + 1} = \frac{y-a}{1-2ay+yy} \times \frac{z}{y^{n-1}} + \frac{y-b}{1-2by+yy} \times \frac{z}{y^{n-1}} + \&c.$ Because $\frac{y^{2n} - 2zy^n + 1}{y^{2n}} = \overline{1 - 2ay^{-1} + y^{-2}} \times \overline{1 - 2by^{-1} + y^{-2}} \times \&c.$ it follows (as in art. 728), that $\frac{n - nzy^n}{y^{2n} - 2zy^n + 1} = \frac{1-ay}{1-2ay+yy} + \frac{1-by}{1-2by+yy} + \&c.$ The sum of those equations gives $\frac{n - nzz}{y^{2n} - 2zy^n + 1} = \frac{\overline{y-a} \times zy^{-n+1} + 1 - ay}{1-2ay+yy} + \frac{\overline{y-b} \times zy^{-n+1} + 1 - by}{1-2by+yy} + \&c.$; and it follows, from art. 772, that the fluent of $\frac{\dot{y}}{y^{2n} - 2zy^n + 1}$ is assignable by circular arks and logarithms with algebraic quantities.

776. By a similar application of what was shown in art. 728, if we suppose $xx - Ax \pm B = \overline{x-a} \times \overline{x-b}$, the fraction $\frac{\dot{x}}{xx - Ax + B}$ is resolved into fractions that shall have the simple divisors $x-a$, $x-b$ for their respective denominators with invariable coefficients. For since $xx - Ax \pm B = \overline{x-a} \times \overline{x-b}$, it follows

follows (art. 728), that $\frac{2x\dot{x}-A\dot{x}}{xx-Ax+B} = \frac{\dot{x}}{x-a} + \frac{\dot{x}}{x-b}$; and because $1 - Ax^{-1} + Bx^{-2} = \overline{1 - ax^{-1}} \times \overline{1 - bx^{-1}}$, it follows that $\frac{Ax\dot{x}-2B\dot{x}}{xx-Ax+B} = \frac{a\dot{x}}{x-a} + \frac{b\dot{x}}{x-b}$. From these two equations we have $\frac{\dot{x}}{xx-Ax+B} = \frac{\frac{1}{2}A-a}{2B-\frac{1}{2}AA} \times \frac{\dot{x}}{x-a} + \frac{\frac{1}{2}A-b}{2B-\frac{1}{2}AA} \times \frac{\dot{x}}{x-b}$ = (because $A = a + b$ and $B = ab$, by the known properties of equations) $\frac{1}{a-b} \times \frac{\dot{x}}{x-a} + \frac{1}{b-a} \times \frac{\dot{x}}{x-b}$. Hence $\frac{1}{xx-Ax+B} = \frac{1}{a-b} \times \frac{1}{x-a} + \frac{1}{b-a} \times \frac{1}{x-b}$. Therefore if $x^3 - Ax^2 + Bx - C = \overline{x-a} \times \overline{x-b} \times \overline{x-c}$, it follows, that $\frac{1}{x^3-Ax^2+Bx-C} = \frac{1}{\overline{x-a} \times \overline{x-b}} \times \frac{1}{x-c}$ = (by what has been shown) $\frac{1}{\overline{a-b} \times \overline{x-a} \times \overline{x-c}} + \frac{1}{\overline{b-a} \times \overline{x-b} \times \overline{x-c}}$ = (by resolving each of these last fractions in the same manner) $\frac{1}{\overline{a-b} \times \overline{a-c} \times \overline{x-a}} + \frac{1}{\overline{a-b} \times \overline{c-a} \times \overline{x-c}} + \frac{1}{\overline{b-a} \times \overline{b-c} \times \overline{x-b}} + \frac{1}{\overline{b-a} \times \overline{c-b} \times \overline{x-c}} = \frac{1}{\overline{a-b} \times \overline{a-c} \times \overline{x-a}} + \frac{1}{\overline{b-a} \times \overline{b-c} \times \overline{x-b}} + \frac{1}{\overline{c-a} \times \overline{c-b} \times \overline{x-c}}$. The continuation of those theorems is manifest, the coefficient of $x - b$, for example, being always the product of the differences $b-a$, $b-c$, &c. by which the root b exceeds the other roots of the equation $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times$ &c. $= o$. This subject is considered by Mr. *Leibnitz, Act. Lips.* 1702, and Mr. *De Moivre, Phil. Trans. N.* 373, &c.

777. But these fractions are briefly discovered in the following manner. Suppose $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} +$ &c. $= \overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} \times$ &c. and let this product be represented by P; let Q represent the product of all the simple divisors, the first $x-a$ excepted; that is, let $Q = \overline{x-b} \times \overline{x-c} \times \overline{x-d}$

$\times \overline{x-d} \times$ &c. Suppose $a, b, c, d,$ &c. to be unequal, and r being any integer positive number less than n, suppose $\frac{x^r}{P}$ or

$$\frac{x^r}{x^n - Ax^{n-1} + Bx^{n-2} - \&c.} = \frac{L}{x-a} + \frac{M}{x-b} + \frac{N}{x-c} + \&c.$$

where L, M, N, &c. represent the invariable coefficients that are to be determined. By reducing those fractions to a common denominator, and multiplying by P or $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times$ &c. we have $x^r = LQ + MQ \times \frac{x-a}{x-b} + NQ \times \frac{x-a}{x-c}$ + &c. Then by supposing $x = a$, or $x-a = 0$, we find that x^r (*i. e.* in this case a^r) is equal to LQ, or that $a^r = L \times \overline{a-b} \times \overline{a-c} \times \overline{a-d} \times$ &c. and $L = \frac{a^r}{\overline{a-b} \times \overline{a-c} \times \overline{a-d} \times \&c.}$. In the same manner by supposing $x = b$, we find $M = \frac{b^r}{\overline{b-a} \times \overline{b-c} \times \overline{b-d} \times \&c.}$ The other coefficients of the fractions into which $\frac{x^r}{P}$ is to be resolved, are expressed by similar values.

778. Because $P = Q \times \overline{x-a}$, it follows, that $\dot{P} = \dot{Q} \times \overline{x-a} + \dot{x}Q$; and when $x = a$, $\dot{P} = \dot{x}Q$, or Q (which in this case is equal to $\overline{a-b} \times \overline{a-c} \times \overline{a-d} \times$ &c.) $= \frac{\dot{P}}{\dot{x}} = na^{n-1} - \overline{n-1} \times Aa^{n-2} + \overline{n-2} \times Ba^{n-3}$ — &c. Therefore $L = \frac{a^r}{Q} = \frac{a^r}{na^{n-1} - \overline{n-1} \times Aa^{n-2} + \overline{n-2} \times Ba^{n-3} - \&c.}$ In the same manner $M = \frac{b^r}{nb^{n-1} - \overline{n-1} \times Ab^{n-2} + \overline{n-2} \times Bb^{n-3} - \&c.}$ and the values of the other coefficients are similar. The rule for

for finding the coefficient in any of those fractions (as in $\frac{N}{x-c}$) into which $\frac{x^r}{x^n - Ax^{n-1} + Bx^{n-2} - \&c.}$ is to be resolved, is, substitute c for x in the numerator x^r, find the fluxion of the denominator $x^n - Ax^{n-1} + \&c.$ which being divided by $\dot{x}$, and c being substituted for x in the quotient, you have the denominator of the value of N.

779. Let it be required to resolve $\frac{x^r}{x^{2n} - Ax^{2n-1} + Bx^{2n-2} - \&c.}$ into fractions that shall have quadratic denominators, where r is supposed to be any integer and positive number less than $2n$; and the denominator (P) to be the product of the quadratic divisors $xx - 2ax + gg$, $xx - 2bx + hh$, $xx - 2cx + kk$, &c.; suppose $\frac{x^r}{P} = \frac{L - lx}{xx - 2ax + gg} + \frac{M - mx}{xx - 2bx + hh} + \frac{N - nx}{xx - 2cx + kk} +$ &c. Let Q represent the product of all the quadratic divisors, the first $xx - 2ax + gg$ excepted; that is, let $P = \overline{xx - 2ax + gg} \times Q$. Then by reducing the fractions to a common denominator, $\overline{L - lx} \times Q + \overline{M - mx} \times Q \times \frac{xx - 2ax + gg}{xx - 2bx + hh} + \overline{N - nx} \times Q \times \frac{xx - 2ax + gg}{xx - 2cx + kk}$ + &c. $= x^r$. Let e and f be the roots of the equation $xx - 2ax + gg = 0$. Let M be the value of Q when $x = e$, and N its value when $x = f$. Then substituting e for x, $xx - 2ax + gg$, and all the terms that are multiplied by it, vanish; consequently $\overline{L - le} \times M = e^r$. In the same manner, by substituting f for x, $\overline{L - lf} \times N = f^r$. Hence $L = \frac{e^r f}{M \times \overline{f - e}} - \frac{f^r e}{N \times \overline{f - e}} =$ (because $ef = gg$) $\frac{gg}{f - e} \times \overline{\frac{e^{r-1}}{M} - \frac{f^{r-1}}{N}}$. Because $P = \overline{xx - 2ax + gg} \times Q$, it follows (by taking the fluxions), that $\dot{P} = \overline{2x - 2a} \times Q\dot{x} + \dot{Q} \times \overline{xx - 2ax + gg}$; and by substituting e for x, $\frac{\dot{P}}{\dot{x}}$ (which in this

this case is $2ne^{2n-1} - \overline{2n-1} \times Ae^{2n-2} + \overline{2n-2} \times Be^{2n-3} - \&c.)$ $= \overline{2e-2a} \times M =$ (because $e+f=2a$, and $e-f = 2e-2a$), $\overline{e-f} \times M$. In the same manner $N \times \overline{f-e} = 2nf^{2n-1} - \overline{2n-1} \times Af^{2n-2} + \&c.$ Therefore $L = \frac{ggf^{r-1}}{2nf^{2n-1} - \overline{2n-1} \times Af^{2n-2} + \&c.} - \frac{gge^{r-1}}{2ne^{2n-1} - \overline{2n-1} \times Ae^{2n-2} + \&c.}$ And in like manner we shall find $l = \frac{-e^r}{M \times \overline{e-f}} - \frac{f^r}{N \times \overline{f-e}} = \frac{-e^r}{2ne^{2n-1} - \overline{2n-1} \times Ae^{2n-2} + \&c.} - \frac{f^r}{2nf^{2n-1} - \overline{2n-1} \times Af^{2n-2} + \&c.}$ The values of the coefficients M and m, N and n, &c. are similar.

780. Let gg, hh, kk, &c. the last terms of the quadratic divisors, be all equal to each other, and to unit. Then $M = \overline{ee-2be+1} \times \overline{ee-2ce+1} \times \overline{ee-2de+1} =$ (because $ee - 2ae + 1 = 0$, and $ee + 1 = 2ae$) $\overline{2ae-2be} \times \overline{2ae-2ce} \times \overline{2ae-2de} \times \&c. = e^{n-1} \times \overline{2a-2b} \times \overline{2a-2c} \times \overline{2a-2d} \times \&c.$ In the same manner, $N = f^{n-1} \times \overline{2a-2b} \times \overline{2a+2c} \times \overline{2a-2d} \times \&c.$ Therefore if I represent the product of the differences by which $2a$ the middle coefficient of the first divisor exceeds $2b$, $2c$, $2d$, &c. the middle coefficients of the other divisors; then $M = e^{n-1}I$ and $N = f^{n-1}I$. Therefore, by substituting those expressions for M and N in the values of L and l in the last article $L = \frac{e^{n-r} - f^{n-r}}{I \times \overline{e-f}}$ when n is greater than r; or $L = \frac{e^{r-n} - f^{r-n}}{I \times \overline{f-e}}$ when n is less than r; that is, the difference of n and r being represented by m, $L = \frac{e^m - f^m}{\mp I \times \overline{e-f}}$ where the sign

of

of I is positive or negative, according as n is greater or less than r. In like manner $l = \dfrac{e^{n-r-1} - f^{n-r-1}}{\text{I} \times \overline{e-f}}$ when $n-1$ is greater than r, or $l = \dfrac{e^{r-n+1} - f^{r-n+1}}{\text{I} \times \overline{e-f}}$ when $n-1$ is less than r; that is, $l = \dfrac{e^{m-1} - f^{m-1}}{\text{I} \times \overline{e-f}}$ in the former case, and $l = \dfrac{e^{m+1} - f^{m+1}}{\text{I} \times \overline{e-f}}$ in the latter. Because e and f are the roots of the equation $xx - 2ax + 1 = 0$, $e = a + \sqrt{aa-1}$, $f = a - \sqrt{aa-1}$, and $e-f = 2\sqrt{aa-1}$, and $\text{L} = \dfrac{\overline{a+\sqrt{aa-1}}^{\,m} - \overline{a-\sqrt{aa-1}}^{\,m}}{\mp\text{I} \times 2\sqrt{aa-1}}$ $= \dfrac{\overline{a+\sqrt{aa-1}}^{\,m} - \overline{a-\sqrt{aa-1}}^{\,m}}{\mp 2\text{I}\sqrt{1-aa}} \times \sqrt{-1}$. Hence if B$b$, the cosine of the ark AB (*fig.* 302) be equal to a, the radius OA being unit, the ark AQ be to the ark AB as m to 1, and Qq be the cosine of AQ, then Ob being equal to $\sqrt{1-aa}$, it follows (by comparing the value of S determined in art. 760), that $\text{L} = \dfrac{\mp\text{O}q}{\text{I}\times\text{O}b}$. Let the ark QZ be made equal to AB, so that AZ may be equal to $\overline{m-1} \times \text{AB}$, or $\overline{m+1} \times \text{AB}$, according as $n-1$ is greater or less than r, and Zz be the cosine of the ark AZ; then $l = \dfrac{\mp\text{O}z}{\text{O}b\times\text{I}}$. Therefore the fraction $\dfrac{\text{L}-lx}{1-2ax+xx} = \dfrac{\mp\text{O}q \mp \text{O}z \times x}{\text{O}b\times\text{I}} \times \dfrac{1}{1-2ax+xx}$. The values of the other fractions are similar. And thus it appears how the fluent of $\dfrac{x^r\dot{x}}{x^{2n} - \text{A}x^{2n-1} + \text{B}x^{2n-2} \text{ -\&c.}}$ is assignable by circular arks and logarithms when the denominator is the product of any quadratic divisors.

781. If the values of L and l are to be expressed algebraically, then raise $a+1$ to the power of the exponent m, the differ-

 ence

ence of n and r, by art. 748; multiply the 2d, 4th, 6th, &c. terms of this power by 1, $aa-1$, $\overline{aa-1}^2$, $\overline{aa-1}^3$, &c. respectively; and the sum of the products divided by $\overline{2a-2b} \times \overline{2a-2c} \times \overline{2a-2d} \times$ &c. will be equal to $+$ L or $-$ L, according as n is greater or less than r. The other coefficient l of the fraction $\frac{L-lx}{1-2ax+xx}$ is found by multiplying the like terms of the power of $a+1$ of the exponent n-r-1 or $r-n+1$ (according as $n-1$ is greater or less than r) by 1, $aa-1$, $\overline{aa\text{-}1}^2$, $\overline{aa\text{-}1}^3$, &c. respectively, dividing the sum of the products by $\overline{2a-2b} \times \overline{2a-2c} \times \overline{2a-2d} \times$ &c. The quotient will be equal to $+l$ in the former case, but to $-l$ in the latter. The coefficients M and m, N and n, &c. are found in like manner. But when $n=r$, the coefficients L, M, N, &c. vanish; and when n-$1=r$, the other coefficients l, m, n, &c. vanish.

782. When the fraction that is to be resolved in this manner is of the form $\frac{1}{1-2zx^n+x^{2n}}$, where the denominator is a trinomial, the coefficients of the fractions $\frac{L-lx}{1-2ax+xx}$, &c. into which it is to be resolved, may be more briefly determined from art. 779, by which (substituting in this case o for r) L $= \frac{e}{N \times \overline{e-f}} - \frac{f}{M \times \overline{e-f}}$; where, P being supposed equal to $\overline{xx-2ax+1} \times Q$, M is the value of Q when $x=e$, and N its value when $x=f$. By taking the fluxions, as in that article, $\dot{P} = 2nx^{2n-1}\dot{x} - 2nzx^{n-1}\dot{x} = \overline{2x\dot{x}-2a\dot{x}} \times Q + \dot{Q} \times \overline{xx-2ax+1}$, and substituting e for x, and M for Q, $2ne^{2n-1} - 2nze^{n-1} = \overline{2e-2a} \times M = \overline{e-f} \times M$; consequently $\overline{e-f} \times M = 2ne^{n-1} \times \overline{e^n-z}$. In the same manner $\overline{f-e} \times N = 2nf^{n-1} \times \overline{f^n-z}$. But $e^{2n} - 2ze^n + 1 = o$ and $e^n + \frac{1}{e^n}$ (or

e^n

$e^n + f^n) = 2l$, and $e^n - f^n = 2e^n - 2l$ or $2l - 2f^n$; so that $\overline{e-f} \times M = ne^{n-1} \times \overline{e^n - f^n}$ and $\overline{e-f} \times N = nf^{n-1} \times \overline{e^n - f^n}$. Therefore $L = \frac{e}{nf^{n-1} \times \overline{e^n - f^n}} - \frac{f}{ne^{n-1} \times \overline{e^n - f^n}} = \frac{1}{n}$. By art. 779, $l = \frac{-1}{M \times \overline{e-f}} - \frac{1}{N \times \overline{f-e}} = \frac{-1}{ne^{n-1} \times \overline{e^n - f^n}} + \frac{1}{nf^{n-1} \times \overline{e^n - f^n}} = \frac{1}{n} \times \frac{e^{n-1} - f^{n-1}}{e^n - f^n} = \frac{1}{n} \times \frac{\overline{a + \sqrt{aa-1}}^{\,n-1} - \overline{a - \sqrt{aa-1}}^{\,n-1}}{\overline{a + \sqrt{aa-1}}^{\,n} - \overline{a - \sqrt{aa-1}}^{\,n}}$. Therefore, by comparing the value of S in art. 760, it appears that if Bb the cosine of the ark AB be equal to a, AK $= n \times$ AB, AZ $= \overline{n-1} \times$ AB, Kk and Zz be the cosines of AK and AZ; then $l = \frac{1}{n} \times \frac{Oz}{Ok}$; and the fraction $\frac{L - lx}{1 - 2ax + xx} = \frac{\frac{1}{n} - \frac{Oz}{n \times Ok} \times x}{1 - 2ax + xx}$, which coincides with Mr. *De Moivre's* fifth proposition, *lib.* 1. *Miscel. Analyt.* as it is concisely expressed, *p.* 42, of that treatise.

783. When the fraction proposed is of this form $\frac{x^r}{1 - 2lx^n + x^{2n}}$ r being any integer positive number less than $2n$, let the difference of n and r be represented by m, as above; and L $=$ (art. 780) $\frac{e^m - f^m}{I \times \overline{e-f}} =$ (because $Ie^{n-1} = M = ne^{n-1} \times \frac{e^n - f^n}{e - f}$ by what was proved in the last article) $\frac{1}{n} \times \frac{e^m - f^m}{e^n - f^n}$. Therefore if the ark AQ be to AB as m to 1, and Qq be the cosine

of AQ, $L = \frac{Oq}{n \times Oi}$; and $l = \frac{e^{m-1} - f^{m-1}}{1 \times \overline{e-f}}$ or $\frac{e^{m+1} - f^{m+1}}{1 \times \overline{e-f}}$ according as n is greater or less than r; that is, $l = \frac{1}{n} \times \frac{e^{m-1} - f^{m-1}}{e^n - f^n}$ or $\frac{1}{n} \times \frac{e^{m+1} - f^{m+1}}{e^n - f^n}$; or, supposing AZ $= \overline{m \mp 1} \times$ AB, and Zz to be the cosine of AZ, $l = \frac{Oz}{n \times Oi}$. Therefore the fraction $\frac{L - lx}{1 - 2ax + xx} = \frac{Oq - Oz \times x}{n \times Oi \times \overline{1 - 2ax + xx}}$. When $r = n$, or to $n - 1$, the numerators of those fractions consist of one term only.

784. It appears, as in art. 773, that the fraction $\frac{1 + px + qx^2 + rx^3 + \&c.}{x^n - Ax^{n-1} + Bx^{n-2} - \&c.}$ is equal to $\frac{K}{x-a} + \frac{L}{x-b} + \frac{M}{x-c} + \frac{N}{x-d} + \&c.$ If a, b, c, d, the roots of the equation $x^n - Ax^{n-1} + Bx^{n-2} - \&c. = 0$ be all unequal, the index of x in the numerator $1 + px + qx^2 + \&c.$ be less than n, and we suppose the coefficients K, L, M, N, &c. respectively equal to the quantities that result when we substitute successively a, b, c, d, &c. for x in $\frac{1 + px + qx^2 + rx^3 + \&c.}{nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3} - \&c.}$ This theorem serves for reducing briefly the fluent of $\frac{1 + px + qx^2 + \&c.}{x^n - Ax^{n-1} + Bx^{n-2} - \&c.} \times \dot{x}$ to logarithms or circular arks.

785. Suppose now that some of the factors of the denominator of the fraction, by which the ordinate is expressed, are equal to each other; and let it be required to find the fluent or area. For example, let it be required to find the fluent of $\frac{1 + px + qx^2 + \&c.}{\overline{x+a}^m \times \overline{x+b}^n} \times \dot{x}$. Suppose $\frac{1 + px + qx^2 + \&c.}{\overline{x+a}^m \times \overline{x+b}^n} =$

$\frac{H}{\overline{x+a}^m} + \frac{K}{\overline{x+a}^{m-1}} + \frac{L}{\overline{x+a}^{m-2}} \ldots + \frac{h}{\overline{x+b}^n} + \frac{k}{\overline{x+b}^{n-1}} + \frac{l}{\overline{x+b}^{n-2}}$ + &c. where H, K, L, h, k, l, &c. are the invariable coefficients that are to be determined, and the fractions of each sort are to be continued till their number be equal to m and n respectively. By reducing the equation to a common denominator, $H \times \overline{x+b}^n + K \times \overline{x+a} \times \overline{x+b}^n + L \times \overline{x+a}^2 \times \overline{x+b}^n \ldots + h \times \overline{x+a}^m + k \times \overline{x+b} \times \overline{x+a}^m + l \times \overline{x+b}^2 \times \overline{x+a}^m + \&c. = 1 + px + qx^2 + rx^3 + \&c.$ By supposing $x + a = 0$, or $x = -a$, all the terms of this equation that involve $x+a$ vanish, and we find $H \times \overline{b-a}^n = 1 - pa + qa^2 - ra^3 + \&c.$ or $H = \frac{1 - pa + qa^2 - \&c.}{\overline{b-a}^n}$. By taking the fluxion of the equation, dividing each term by $\dot{x}$, and then supposing $x = -a$, we have $nH \times \overline{b-a}^{n-1} + K \times \overline{b-a}^n = p - 2qa + 3ra^2 - \&c.$ By taking the fluxions again, dividing by $\dot{x}$, and supposing $x = -a$, we find $n \times \overline{n-1} \times H \times \overline{b-a}^{n-2} + 2nK \times \overline{b-a}^{n-1} + 2L \times \overline{b-a}^n = 2q - 6ra +$ &c. Thus the values of H, K, L, &c. are easily computed; and by proceeding in like manner, and supposing $x+b=0$, the values of h, k, l, &c. are determined. If the fraction proposed be $\frac{1}{\overline{x+a}^m \times \overline{x+b}^n}$, that is, if p, q, r, &c. be supposed to vanish, then $H = \frac{1}{\overline{b-a}^n}$, $K = \frac{-n}{\overline{b-a}^{n+1}}$, $L = n \times \frac{n+1}{2} \times \frac{1}{\overline{b-a}^{n+1}}$, $h = \frac{1}{\overline{a-b}^m}$, $k = \frac{-m}{\overline{a-b}^{m+1}}$, $l = m \times \frac{m+1}{2} \times \frac{1}{\overline{a-b}^{m+1}}$, &c. which coincides with Mr. *Leibnitz*'s theorem, *Act. Lips.* 1703. If

the fraction proposed be $\frac{1}{\overline{x+a}^m \times \overline{x+b}^n \times \overline{x-c}^s}$, and we suppose it equal to $\frac{H}{\overline{x+a}^m} + \frac{K}{\overline{x+a}^{m-1}} + \frac{L}{\overline{x+a}^{m-2}} +$ &c. it will appear in the same manner, that $H = \frac{1}{\overline{b-a}^n \times \overline{c-a}^s}$, $K = \frac{-mH}{b-a} - \frac{sH}{c-a}$,

$$2C = \overline{\frac{m \times \overline{m-1}}{\overline{b-a}^2} + \frac{2ms}{\overline{b-a} \times \overline{c-a}} + \frac{s \times \overline{s-1}}{\overline{c-a}^2}} \times H + \overline{\frac{2m}{b-a} - \frac{2s}{c-a}}$$

$\times$ K, &c.

786. The ordinate or fraction $\frac{1 + px + qx^2 + \&c.}{\overline{x+a}^m \times \overline{x+b}^n}$, being resolved in this manner, the fluent will be found (by art. 737) equal to $\frac{-H}{\overline{m-1} \times \overline{x+a}^{m-1}} \frac{-K}{\overline{m-2} \times \overline{x+a}^{m-2}} \cdots \frac{-h}{\overline{n-1} \times \overline{x+b}^{n-1}}$ $\frac{-k}{\overline{n-2} \times \overline{x+b}^{n-2}}$, &c. If the last coefficient of each sort vanish, the fluent will be assignable in algebraic terms: in other cases, the fluent is assigned by logarithms with algebraic quantities.

787. If we suppose the fraction $\frac{1 + px + qx^2 + \&c.}{\overline{x+a}^m \times \overline{x+b}} =$ $\frac{K}{x+b} + \frac{A + Bx + Cx^2 \ldots Gx^{m-1}}{\overline{x+a}^m}$, that it may be resolved after Mr. *De Moivre's* method, *Miscel. Analyt. p.* 59; then $K \times \overline{x+a}^m + Ab + \overline{A+Bb} \times x + \overline{B+Cb} \times x^2 + \overline{C+Db} \times x^3 = 1 + px + qx^2 +$ &c. By supposing $x + b = o$, or $x = -b$, $K \times \overline{a-b}^m = 1 - pb + qb^2 -$ &c. The supposition of $x = o$ gives $Ka^m + Ab = 1$. By taking the fluxions of the equation, dividing by $\dot{x}$, and supposing $x = o$, nKa^{m-1}

+

$+ A + Bb = p$; and by proceeding in this manner the coefficients K, A, B, C, &c. may be determined. If the fraction be $\frac{1}{\overline{x+a}^m \times \overline{x+b}}$, the coefficient of any term in the numerator of the second fraction, as of Fx^r, is found by raising $a-b$ to the power of the exponent m, rejecting as many of the terms of this power a^m, $ma^{m-1}b$, &c. as there are units in $r+1$; and dividing the sum of these that remain by $b^{r+1} \times \overline{a-b}^m$; for the quotient will give $+F$ or $-F$, according as r is an even or odd number. The fraction $\frac{1}{\overline{1-ax}^m \times \overline{1-bx}}$ is resolved in like manner by a similar rule, and supposing it equal to $\frac{K}{1-bx}$ $+ \frac{A + Bx + Cx^2 \ldots Gx^{m-1}}{\overline{1-ax}^m}$; $K = \frac{b^m}{\overline{b-a}^m}$, $A = 1 - \frac{b^m}{\overline{b-a}^m}$

$B = bA + \frac{mab^m}{\overline{b-a}^m}$, $C = bB - \frac{m \times \overline{m-1} \times b^m a^2}{2 \times \overline{b-a}^m}$, &c.

788. The fluxion $\frac{x^{\frac{m}{r}} \dot{x}}{e+fx^n}$ is transformed into $\frac{rz^{m+r-1} \dot{z}}{e+fz^{rn}}$ by supposing $x^{\frac{1}{r}} = z$; because $x^{\frac{m+r}{r}} = z^{m+r}$, $x^{\frac{m}{r}} \dot{x} = rz^{m+r-1} \dot{z}$ and $x^n = z^{rn}$. In like manner the fluxion is transformed, so as to become rational, when the denominator is a trinomial; and the fluent may be found by the preceding articles.

789. Sir *Isaac Newton* has given some excellent theorems for reducing fluents to others of a more simple form, in the 7th, 8th, and 11th *prop.* of his treatise *De Quadrat. Curvar.* Let R $= e + fx^n + gx^{2n} + hx^{3n} +$ &c. and, consequently, $\dot{R} =$ $nfx^{n-1}\dot{x} + 2ngx^{2n-1}\dot{x} + 3nhx^{3n-1}\dot{x} +$ &c. Let $\dot{A} =$ $x^{m-1}\dot{x}$.

$x^{m-1}\dot{x}R^l$, $\dot{B}=\dot{A}x^n$, $\dot{C}=\dot{B}x^n$, $\dot{D}=\dot{C}x^n$, &c. and $ln+n=p$. Then $me\dot{A}+\overline{p+m}\times f\dot{B}+\overline{2p+m}\times g\dot{C}+\overline{3p+m}\times h\dot{D}+$ &c. $=x^m R^{l+1}$. This theorem will appear by taking the fluxion of $x^m R^{l+1}$, which (by art. 725) is $x^{m-1}R^l\times\overline{mR\dot{x}+\overline{l+1}\times x\dot{R}}=x^{m-1}\dot{x}R^l\times\overline{me+mfx^n+mgx^{2n}+\&c.+\overline{l+1}\times nfx^n\times\overline{l+1}\times 2ngx^{2n}+\&c.}=me\dot{A}+\overline{p+m}\times f\dot{B}+\overline{2p+m}\times g\dot{C}+\overline{3p+m}\times h\dot{D}+$ &c. Let the number of terms in $e+fx^n+gx^{2n}+$ &c. the value of R be represented by q; and if as many of the successive areas A, B, C, D, &c. be known as there are units in $q-1$, the rest can be computed from these, by this theorem. Thus if R be a binomial $e+fx^n$, any one of the areas, A, B, C, D, &c. being given, the rest may be computed from it; and when R is a trinomial $e+fx^n+gx^{2n}$, any two of those areas, as A and B, are sufficient for determining the rest.

790. Let $\dot{H}=x^{m-1}\dot{x}R^{l+1}=x^{m-1}\dot{x}R^l\times\overline{e+fx^n+gx^{2n}+\&c.}=e\dot{A}+f\dot{B}+g\dot{C}+h\dot{D}+$ &c.; and it follows that $H=eA+fB+gC+hD+$ &c. Hence it appears that m and l being any numbers whatsoever, if r and s be any integer numbers, and as many of the areas A, B, C, D, &c. be known as there are units in $q-1$, the fluent of $x^{m+rn-1}\dot{x}\times\overline{e+fx^n}^{l+s}$ may be computed from them.

791. In like manner it appears, that if $R=e+fx^n+gx^{2n}+$ &c. $S=E+Fx^n+Gx^{2n}+$ &c. $\dot{A}=x^{m-1}\dot{x}R^lS^k$, $\dot{B}=\dot{A}x^n$, $\dot{C}=\dot{B}x^n$, &c. and $ln+n=p$, $kn+n=q$, then $x^mR^{l+1}S^{k+1}=meEA+\overline{meF+mfE+pfE+qeF}\times B+\overline{meG+mgE+mfF+pfF+qfF+2pgE+2qeG}\times C+$ &c.

792. From

792. From these, particular theorems are easily deduced that may be of use in the resolution of problems. Let R be the binomial $e-fx^n$; and, as in art. 789, let $\dot{A}=x^{m-1}\dot{x}\,R^l$, $\dot{B}=\dot{A}\,x^n$, $\dot{C}=\dot{B}\,x^n$, $\dot{D}=\dot{C}x^n$, &c.; and $M=ln+n+m$. Let m and $l+1$ be any positive numbers whatsoever, that $x^m\,R^{l+1}$ may vanish either when $x=o$, or $e-fx^n=o$; and let r be any integer positive number; then if A represent the fluent of $x^{m-1}\dot{x}\times\overline{e-fx^n}^l$ that is generated while x flows, and from being o becomes equal to $\overline{\frac{e}{f}}\Big|^{\frac{1}{n}}$, the fluent of $x^{m+rn-1}\dot{x}\times\overline{e-fx^n}^l$ generated in the same time will be to $A\times\frac{e^r}{f^r}$ as $\frac{m}{M}\times\frac{m+n}{M+n}\times\frac{m+2n}{M+2n}\times\frac{m+3n}{M+3n}\times$ &c. to 1; where the fractions $\frac{m}{M}$, &c. are to be continued till their number be equal to r. For in the present case $meA-MfB=o$, by art. 789, and $B=\frac{m}{M}\times\frac{eA}{f}$, $C=\frac{m+n}{M+n}\times\frac{eB}{f}=\frac{m}{M}\times\frac{m+n}{M+n}\times\frac{eeA}{ff}$, $D=\frac{m+2n}{M+2n}\times\frac{eC}{f}=\frac{m}{M}\times\frac{m+n}{M+n}\times\frac{m+2n}{M+2n}\times\frac{e^3A}{f^3}$, and so on. The same theorem serves when $\dot{A}=x^{m-1}\dot{x}\times\overline{fx^n-e}^l$.

793. For example, let $\dot{A}=\frac{\dot{x}}{\sqrt{1-xx}}$, and consequently A equal to the fourth part of the circumference of the circle whose radius is 1, and the fluent of $\frac{x^{2r}\dot{x}}{\sqrt{1-xx}}$ generated while x flows, and from being o becomes equal to 1, will be to A as $\frac{1\times3\times5\times7\times\&c.}{2\times4\times6\times8\times\&c.}$ to 1; where the fractions are to be continued till their number be equal to r. Because in this case $\dot{A}$ or $x^{m-1}\dot{x}\times\overline{e-fx^n}^l=x^{1-1}\dot{x}\times\overline{1-xx}^{-\frac{1}{2}}$, so that $m=1$, $l=$

$l = -\frac{1}{2}$, $n = 2$, $M = 2$, $e = 1$, and $f = 1$. The fluxion $\frac{\dot{x}}{\sqrt{1-xx}}$ is transformed into $\frac{\dot{z}}{1+zz}$, by supposing $x = \frac{z}{\sqrt{1+zz}}$, and $\frac{x^{2r}\dot{x}}{\sqrt{1-xx}}$ into $\frac{\dot{z}z^{2r}}{\overline{1+zz}^{r+1}}$; the fluent of which is, therefore, to A the fluent of $\frac{\dot{z}}{1+z^2}$ (or the quadrantal ark of the circle of the radius 1) as $\frac{1 \times 3 \times 5 \times 7}{2 \times 4 \times 6 \times 8} \times$ &c. to 1. If we suppose $\dot{A} = \dot{x}\sqrt{1-xx}$, in which case A is the fourth part of the area of the circle whose radius is 1, then the fluent of $x^{2r}\dot{x}\sqrt{1-xx}$ will be to A as $\frac{1 \times 3 \times 5 \times 7}{4 \times 6 \times 8 \times 10} \times$ &c. to 1. By supposing $x = \frac{z}{\sqrt{1+zz}}$, the corresponding fluent of $\frac{z^{2r}\dot{z}}{\overline{1+zz}^{r+2}}$ will be to the fluent of $\frac{\dot{z}}{\overline{1+zz}^{2}}$, in the same ratio. In like manner other theorems of this kind may be deduced from those in art. 789, &c.

794. The fluxion $x^{m-1}\dot{x} \times \overline{e + fx^n}^{l}$ being transformed, as in art. 742, by supposing $e + fx^n = z$, the fluent will be measured by the areas of conic sections when $\frac{m}{n}$ is any integer number positive or negative, by art. 789. When $\frac{m}{n} + l$ is any integer number, the same will appear by supposing $z = \frac{e+fx^n}{x^n}$. Or if l be equal to the fraction $\frac{q}{s}$, we may suppose $z = \overline{e+fx^n}^{\frac{1}{s}}$ in the former case, and $z = \overline{\frac{e+fx^n}{x^n}}^{\frac{1}{s}}$ in the latter. The fluxion $\dot{x}x^{rn-1} \times \overline{\frac{e+fx^n}{g+hx^n}}^{l}$ is transformed,

by

by supposing $\frac{e+fx^n}{g+hx^n} = z$ (and consequently $x^n = \frac{e-gz}{hz-f}$ and $\frac{n\dot{x}}{x} = \frac{gf-eh}{\overline{e-gz} \times \overline{hz-f}} \times \dot{z}$) into $\frac{gf-eh}{n} \times z^l \dot{z} \times \frac{\overline{e-gz}^{r-1}}{\overline{hz-f}^{r+1}}$. By supposing $z = gx^n + \frac{1}{2}f$, the fluxion $x^{rn-1}\dot{x} \times \overline{e+fx^n+gx^{2n}}^{\mp\frac{1}{2}}$ is transformed into $\frac{\dot{z}}{n} \times \overline{\frac{2z-f}{2g}}\Big|^{r-1} \times \overline{e-\frac{ff}{g}+zz}\Big|^{\mp\frac{1}{2}}$; and the fluent may be found in both those cases by the preceding articles, when r is any integer number. If we suppose $y = \frac{\sqrt{e+fx^n+gx^{2n}} - \sqrt{e},}{x^n}$ and transform the last fluxion from x to y, its expression will become rational, as is shown, *Miscel. Analyt. p.* 65. When any of those fluxions is multiplied or divided by a rational binomial $E + Fx^n$, or trinomial $E + Fx^n + Gx^{2n}$, or by any quantity that can be resolved into such binomial or trinomial factors, the fluent may be measured by the areas of conic sections (that is, either by algebraic quantities, or by circular arks, or logarithms, or these compounded together), by the preceding articles.

795. When a fluxion is proposed that involves an irrational quantity, the fluent is sometimes obtained in finite terms, or compared with a circular ark or logarithm, by supposing the quantity that is under the radical sign equal to a new flowing quantity. Thus if $\dot{Q} = \frac{\dot{x}}{\overline{E+Fx} \times \overline{e+fx}} \times \overline{\frac{E+Fx}{e+fx}}\Big|^{\frac{m}{n}}$, and we suppose $z = \frac{E+Fx}{e+fx}$, then $\frac{\dot{z}}{z} = \frac{Fe-Ef}{\overline{E+Fx} \times \overline{e+fx}} \times \dot{x}$, and $\dot{Q} = \frac{\dot{z}z^{\frac{m}{n}-1}}{Fe-Ef}$; consequently the fluent $Q = \frac{nz^{\frac{m}{n}}}{m \times \overline{Fe-Ef}} = \frac{n}{m \times \overline{Fe-Ef}} \times \overline{\frac{E+Fx}{e+fx}}\Big|^{\frac{m}{n}}$. But this is often more easily obtained

ed by transforming the fluxion from the sine or cosine of an ark to the tangent or secant, or to the sum or difference of the secant and tangent, or by the converse operations. If we suppose $x = \frac{z}{\sqrt{1+zz}}$ (z being the tangent that corresponds to the sine x), then $\frac{\dot{x}}{\sqrt{1-xx}} = \frac{\dot{z}}{1+zz}$. Hence if $\dot{Q} = \frac{-\dot{x}}{x^2\sqrt{1-xx}}$ then $\dot{Q} = \frac{-\dot{z}}{z^2}$, and $Q = \frac{1}{z} = \frac{\sqrt{1-xx}}{x}$. And if $\dot{Q} = \frac{a\dot{x}}{aa\mp xx\sqrt{1-xx}} = \frac{a\dot{z}}{aa+\overline{aa-1}\times zz}$, then Q is equal to the ark of a circle described with the radius $\frac{a}{\sqrt{aa\mp 1}}$ that has its tangent equal to z or $\frac{x}{\sqrt{1-xx}}$. If we suppose $x + \sqrt{xx\mp 1} = z$, then $\frac{\dot{x}}{\sqrt{xx\mp 1}} = \frac{\dot{z}}{z}$. If we suppose $\frac{a+\sqrt{aa\pm xx}}{x} = z$, then $\frac{-a\dot{x}}{x\sqrt{aa\pm xx}} = \frac{\dot{z}}{z}$; so that the fluent is the logarithm of z, the *modulus* being 1. And by supposing $\frac{e+\sqrt{ee+ffx^{2n}}}{x^n} = z$, $\frac{-\dot{x}}{x\sqrt{ee+ffx^{2n}}}$ is transformed into $\frac{\dot{z}}{nez}$; so that the fluent is $\frac{1}{ne} \times$ log. z, the *modulus* being unit.

796. Supposing, as in art. 789, $R = e+fx^n$, $\dot{A} = x^{m-1}\dot{x}\, R^l$, and $\dot{B} = \dot{A}x^n$ we found $meA + MfB = x^m R^{l+1}$; from which it follows, that if neither $m=o$, nor $M=o$, A and B depend mutually upon each other; but if $m = o$, B is assignable in finite algebraic terms; and if $M = o$, A is assignable in such terms. If neither $\frac{m}{n}$ nor $\frac{M}{n}$ be equal to o, or to an integer number, the fluents of all the fluxions in the series

$\dot{A}\, x^{2n}$,

$\dot{A}x^{2n}$, $\dot{A}x^{n}$, $\dot{A}$, $\dot{A}x^{-n}$, $\dot{A}x^{-2n}$, &c. (which may be continued either way) depend upon the fluent of any one fluxion in the series; but when either $\frac{m}{n}$ or $\frac{M}{n}$ is an integer, or when either of them vanishes, this cannot be said of the whole series. Let $\dot{A} = \frac{a\dot{x}}{x^2\sqrt{aa-xx}}$, where $M = 0$, $m = -1$, and $A = \frac{\sqrt{aa-xx}}{ax}$, but the fluent of $\dot{A}x^2$ $(= \frac{a\dot{x}}{\sqrt{aa-xx}})$ is the circular ark whose sine is x, the radius being a: the fluents of $\dot{A}x^{-2}$, $\dot{A}x^{-4}$, &c. depend upon the former, and are assignable in finite algebraic terms; but the fluents of $\dot{A}x^4$, $\dot{A}x^6$, &c. depend upon the latter, and are assignable by that circular ark with algebraic quantities. If $\dot{A} = \frac{-x\dot{x}}{\sqrt{aa-xx}}$, $m = 0$, $M = 1$, $A = \sqrt{aa-xx}$, and the fluents of $\dot{A}x^2$, $\dot{A}x^4$, &c. are assignable by algebraic quantities; but the fluent of $\dot{A}x^{-2}$ $(= \frac{-\dot{x}}{x\sqrt{aa-xx}})$ is the logarithm of $\frac{a+\sqrt{aa-xx}}{x}$, the *modulus* being unit, and the fluents of $\dot{A}x^{-4}$, $\dot{A}x^{-6}$, &c. depend upon this logarithm. In like manner, if $\dot{A} = \frac{\dot{x}}{\sqrt{xx-1}}$, A is the logarithm of $x + \sqrt{xx-1}$, and the fluents of $\dot{A}x^2$, $\dot{A}x^4$, &c. depend upon it; but the fluents of $\dot{A}x^{-2}$, $\dot{A}x^{-4}$, &c. are assignable in finite algebraic terms. If $\dot{A} = \frac{x\dot{x}}{\sqrt{xx-1}}$, $A = \sqrt{xx-aa}$, and the fluents of $\dot{A}x^2$, $\dot{A}x^4$, &c. are assignable in finite algebraic terms; but the fluent of $\dot{A}x^{-2}$ $(= \frac{\dot{x}}{x\sqrt{xx-1}})$ is the ark whose secant is x, the radius being unit, and the fluents of $\dot{A}x^{-4}$, $\dot{A}x^{-6}$, &c. depend

depend upon it. If $\dot{A} = \frac{x^{m-1}\dot{x}}{\sqrt{\mp aa \mp xx}}$, and m be a fraction, the fluents of all the fluxions in the series $\dot{A}x^{\mp 2}$, $\dot{A}x^{\mp 4}$, $\dot{A}x^{\mp 6}$, &c. depend upon A.

797. Let $R = fx^n - e$, $\dot{A} = x^{m-1}\dot{x}R^l$, $\dot{B} = \dot{A}x^{-n}$, $\dot{C} = \dot{B}x^{-n} = \dot{A}x^{-2n}$, $\dot{D} = \dot{C}x^{-n} = \dot{A}x^{-3n}$, &c. and $M = ln + n + m$, as formerly; then when $fx^n = e$, $B = \frac{M-n}{m-n} \times \frac{fA}{e}$, $C = \frac{M-2n}{m-2n} \times \frac{fB}{e}$, $D = \frac{M-3n}{m-3n} \times \frac{Cf}{e}$ &c. Therefore r being any integer positive number, if $\dot{Q} = \dot{A}x^{-rn}$, $Q : \frac{Af^r}{e} :: \frac{M-n}{m-n} \times \frac{M-2n}{m-2n} \times \frac{M-3n}{m-3n} \times$ &c. (where these fractions are to be continued till their number be equal to r) : 1. For example, let $\dot{A} = \frac{\dot{x}}{x\sqrt{xx-1}}$, $\dot{Q} = \frac{\dot{x}}{x^{2r+1}\sqrt{xx-1}}$, then $Q : A :: \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} \times$ &c. : 1. these fluents are generated while $\frac{1}{x}$ from being o becomes equal to 1.

798. After the fluents that can be accurately assigned in finite terms by common algebraic expressions, and those which can be reduced to circular arks and logarithms, the fluents that deserve the next place are such as are assigned by hyperbolic and elliptic arks; which with the former are all comprehended under these which are measured by the lines that bound the conic sections (the triangle and circle being figures of this kind), as the first two are measured by the areas of conic sections. The fluent of $\frac{\dot{x}}{\sqrt{1 \mp x}}$ is of the first class; that of $\frac{\dot{x}}{\sqrt{x} \times \sqrt{1 \mp x}}$ or of $\frac{\dot{x}}{\sqrt{1 \mp xx}}$ is of the second; but the fluents of $\frac{\dot{x}\sqrt{x}}{\sqrt{1 \mp xx}}$, $\frac{\dot{x}}{\sqrt{x} \times \sqrt{1 \mp xx}}$, $\frac{\dot{x}}{\overline{1 \mp xx}|^{\frac{1}{4}}}$ and $\frac{\dot{x}}{\overline{1 \mp xx}|^{\frac{3}{4}}}$ are of the third class, and (as far as has ap-

appeared hitherto) cannot be reduced to the former. The fluents of this class are sometimes required in the resolution of useful problems, and our design obliges us to give some account of them likewise.

799. Let AEH(*fig.*305) be an equilateral hyperbola, that has its centre in S and vertex in A, AD a right line perpendicular to SA, suppose SA$=1$, SN$=x$, and let a circle described with the radius SN from the centre S meet AD in M, let SE bisect the angle ASM, and meet the hyperbola in E; then the hyperbolic ark AE shall be equal to the fluent of $\frac{\dot{x}\sqrt{x}}{2\sqrt{xx-1}}$. For let the ark AE $= s$, SE$= r$, and SP be perpendicular on EP the tangent of the hyperbola in P; then the triangles SMA and SEP will be similar, by art. 181, and $\dot{s} : \dot{r}$: : SE : EP : : SM : AM : : $x : \sqrt{xx-1}$; but SA, SE, and SM, are in continued proportion, or $r = \sqrt{x}$, so that $\dot{r} : \dot{x} :: 1 : 2\sqrt{x}$; consequently $\dot{s} = \frac{\dot{x}\sqrt{x}}{2\sqrt{xx-1}}$; and supposing the fluent of $\frac{\dot{x}\sqrt{x}}{\sqrt{xx-1}}$ to begin to be generated when $x=1$, and thereafter to increase while x increases, it will be always equal to 2AE. If Am be perpendicular to SM in m, and we now suppose S$m=x$, then the hyperbolic ark AE will be the fluent of $\frac{-\dot{x}}{2x\sqrt{x} \times \sqrt{1-xx}}$ (as will appear by substituting in the former fluxion x^{-1} for x); and EP — AE the excess of the tangent above the hyperbolic ark AE will be the fluent of $\frac{-\dot{x}\sqrt{x}}{2\sqrt{1-xx}}$; because EP will then be equal to $\sqrt{\frac{1}{x}-x}$, and its fluxion to $\frac{-\dot{x}-xx\dot{x}}{2x\sqrt{x} \times \sqrt{1-xx}}$.

800. Let AB be perpendicular from A the vertex of the hyperbola to the asymptote SB in B. Suppose now SB $= 1$, upon BA take BL$=x$, join SL, and let it meet the hyperbola in E; from the centre S describe the ark AQ, intersecting SE in Q; and the hyperbolic ark AE shall be equal to the fluent

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of $\frac{\mp\dot{x}\sqrt{1+xx}}{2x\sqrt{x}}$; because if A*b*, LZ, and EK, be perpendicular to the other asymptote in *b*, Z, and K, respectively, S*b* . SK : : EK : A*b* (= LZ) : : SK : SZ, SZ = BL = x, SK = $\sqrt{Sb \times Sz}$ = $\sqrt{x}$, $SE^2 = SK^2 + EK^2 = x + \frac{1}{x}$; and the fluxion of AE being to the fluxion of SK as SE to SK, it is therefore equal to $\frac{\mp\dot{x}}{2x} \times \sqrt{x+\frac{1}{x}}$ or $\frac{\dot{x}\sqrt{1+xx}}{2x\sqrt{x}}$. The fluxion of SE or of QE is $\frac{xx\dot{x}-\dot{x}}{2x\sqrt{x}\times\sqrt{xx+1}}$, by adding which to the fluxion of AE, it appears that AE + EQ is the fluent of $\frac{\dot{x}\sqrt{x}}{\sqrt{1+xx}}$ which begins to be generated when $x = 1$ (or when BL = BA), and thereafter increases while x increases. In the same manner AE—EQ is the fluent of $\frac{-\dot{x}\sqrt{x}}{\sqrt{1+xx}}$ that begins to be generated when $x = 1$, and thereafter increases while x decreases.

801. Suppose SA = 1, AM = x, and 2AE will be the fluent of $\frac{\dot{x}}{\overline{1+xx}|^{\frac{3}{4}}}$ that vanishes with x; as appears by substituting in the first value of $\dot{s}$, in art. 799, $\sqrt{1+xx}$ in the place of x. Suppose SA = 1, Am = x, and 2EP — 2AE will be the fluent of $\frac{-\dot{x}}{\overline{1-xx}|^{\frac{3}{4}}}$ that begins to be generated when $x = 1$, and thereafter increases while x decreases. If we suppose SB = 1, SL = x, then AE ∓ EQ will be the fluent of $\mp\frac{\dot{x}}{\overline{xx-1}|^{\frac{3}{4}}}$ that begins to be generated when $xx = 2$.

802. As for the fluent of $\frac{\dot{x}}{\sqrt{x}\times\sqrt{1\mp xx}}$ or of $\frac{\dot{x}}{\overline{1\mp xx}|^{\frac{3}{4}}}$, it does not appear that it is possible to represent it by any hyperbolic arch and algebraic quantities. But by assuming an elliptic ark, likewise, it may be assigned by the following construction. The rest remaining as in art. 799. Let an ellipse ARD be described having its centre in S, SF the distance of the focus F from

from the centre S equal to the shorter semi-axis SA, and consequently the semi-transverse axis SD : SA : : $\sqrt{2}$: 1. Suppose SA $= 1$, Sm $= x$, take SX upon SA equal to SP (or to a mean proportional betwixt SA and Sm), let the ordinate XR meet the ellipse in R; and the fluent of $\frac{-\dot{x}}{2\sqrt{x}\times\sqrt{1-xx}}$ that begins to be generated when $x = 1$, and thereafter increases while x decreases, will be equal to AR + AE — EP, the difference by which the sum of the elliptic and hyperbolic arks AR and AE exceeds EP the tangent of the latter. For SX $= \sqrt{x}$, and if RT the tangent of the ellipse at R meet SA in T, ST $= \frac{1}{\sqrt{x}}$, XT $=$ ST — SX $= \frac{1-x}{\sqrt{x}}$, XR^2 $= 2 \times \overline{1-x}$, RT^2 $=$ XT^2 + XR^2 $= \frac{1-xx}{x}$; and the fluxion of the elliptic ark AR will be to the fluxion of SX as RT to XT, that is, as $\sqrt{1-xx}$ to $1 - x$, or as $1 + x$ to $\sqrt{1-xx}$; consequently (the fluxion of SX being $\frac{\dot{x}}{2\sqrt{x}}$) the fluxion of the ark AR is

$$\frac{-\dot{x}}{2\sqrt{x}}\times\frac{1+x}{\sqrt{1-xx}}=\frac{-\dot{x}}{2\sqrt{x}\times\sqrt{1-xx}}-\frac{\dot{x}\sqrt{x}}{2\sqrt{1-xx}};$$

and the fluent of $\frac{-\dot{x}}{2\sqrt{x}\times\sqrt{1-xx}}$ (by the latter part of art. 799) equal to AR + AE — EP. If we suppose Am $= x$, AR + AE — EP will be the fluent of $\frac{\dot{x}}{2\times\overline{1-xx}|^{\frac{3}{4}}}$; as will appear by substituting $\sqrt{1-xx}$ for x in the former fluxion. By supposing BL $= z$, and SB $= 1$, the same difference AR + AE — EP gives the fluent of $\frac{\dot{z}}{\sqrt{z}\times\sqrt{1+zz}}$; because if S$n$ $= x$, then $x = \frac{2\sqrt{2\times z}}{1+zz}$. It is likewise the fluent of $\frac{\dot{z}}{\overline{zz-1}|^{\frac{3}{4}}}$, if we suppose SL$=z$, and SB $= 1$, or of $\frac{\dot{z}}{2\times\overline{1+zz}|^{\frac{3}{4}}}$, if we suppose AM $= z$, and SA $= 1$.

803. The

803. The fluent of $\frac{-\dot{x}}{2\sqrt{x} \times \sqrt{1-xx}}$ (which we found equal to AR + AE — EP, art. 802) is equal to AP the ark of the curve that is the *locus* of P (where the perpendiculars from the centre S intersect the tangents of the equilateral hyperbola) which is called the *lemniscata*. For (art. 212) the fluxion of the curve AP is to the fluxion of SP as SE to EP, or as SA to A*m*, that is (supposing SA $=$ 1, and S*m* $= x$), as 1 to $\sqrt{1-xx}$; but SP : SA : : SA : SE, and SE $= \frac{1}{\sqrt{x}}$; consequently SP $= \sqrt{x}$, the fluxion of SP is $\frac{\dot{x}}{2\sqrt{x}}$ and the fluxion of AP $= \frac{-\dot{x}}{2\sqrt{x} \times \sqrt{1-xx}}$. If F be the focus of the hyperbola, FH (*fig.* 306) perpendicular to the tangent EP in H, then it is known that H will be always found in a circle described from the centre S with the radius SA; and if FH produced meet this circle in *h*, SP will be equal to $\frac{1}{2}$ H*h*. From which it appears, that the *lemniscata* may be constructed in the following easy manner. Bisect SF (*fig.* 307) in *f*, from the centre *f* describe a circle with a radius equal to $\frac{1}{2}$ SA, let any right line SX meet this circle in X and *x*, set off SP from S on the same right line always equal to the chord X*x*, and the point P shall be in the *lemniscata*: and the fluents that were described in art. 802 may either be represented by AR + AE—EP, or by the ark AP of this curve, which is so easily constructed.

804. Let AEH be any other hyperbola, SA the semi-transverse and SD the semi-conjugate axis, SA $= a$, SD $= b$, $e = \frac{aa-bb}{2a}$, SE $= r$, SP $= p$, AE $= s$, and $x = \frac{abb}{pp}$; then the hyperbolic ark AE shall be equal to the fluent of $\frac{\dot{x}\sqrt{xa}}{2\sqrt{xx+2ex-bb}}$. For let SH be the semidiameter conjugate to SE, then SE^2 — SH$^2 = aa - bb = 2ea$, or SH$^2 = rr - 2ea$; and SH $\times$ SP $=$ SA $\times$ SB $= ab$, and $rr - 2ea = \frac{aabb}{pp} = ax$. But $\dot{s} : \dot{r} : :$

SE :

3. P. 639.

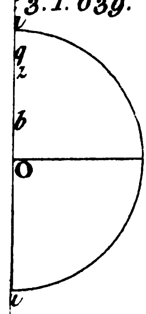
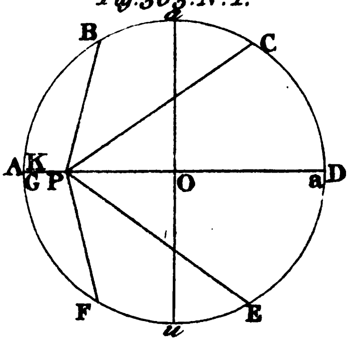


Fig. 303. N°. 1.



ig. 304.

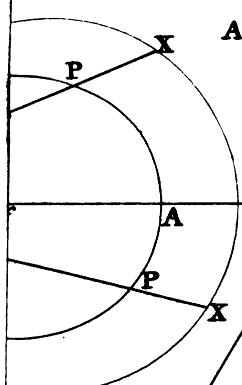
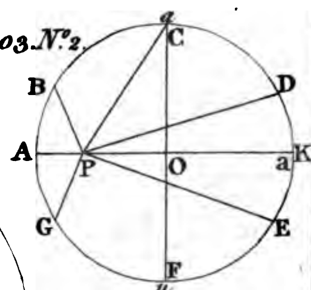


Fig. 303. N°. 2.



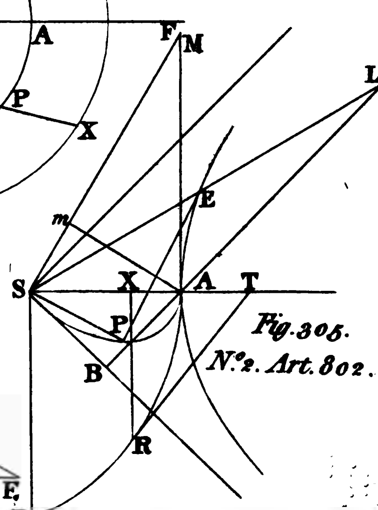
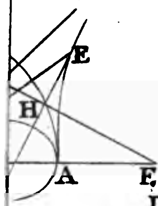


Fig. 305.
N°. 2. Art. 802.



SE : EP : : $\dot{r}$: $\sqrt{rr-pp}$, and $\dot{s} = \frac{r\dot{r}}{\sqrt{rr-pp}} = \frac{\dot{x}\sqrt{xa}}{2\sqrt{xx+2ex-bb}}$.

Because EP $= \sqrt{rr-pp} = \sqrt{\frac{axx+2eax-bb}{x}}$, the fluxion of EP is $\frac{\sqrt{a}}{2x\sqrt{x}} \times \frac{xx\dot{x}+bb\dot{x}}{\sqrt{xx+2ex-bb}}$, and EP — AE (the excess of the tangent above the hyperbolic ark) is the fluent of $\frac{bb\dot{x}\sqrt{a}}{2x\sqrt{x} \times \sqrt{xx+2ex-bb}}$, or (supposing $z = \frac{pp}{a} = \frac{bb}{x}$) of $\frac{-\dot{z}\sqrt{az}}{\sqrt{bb+2ez-zz}}$. It appears likewise that the ark AE is the fluent of $\frac{-a^2b^2\dot{p}}{pp\sqrt{a^2b^2+2aepp-p^4}}$, and that EP—AE is the fluent of $\frac{-p^2\dot{p}}{\sqrt{a^2b^2+2aep^2-p^4}}$. In like manner it appears that if AEB (*fig.* 309) be an ellipsis, S the centre, SA$=a$, SB$=b$, $aa+bb=2ea$, SP be perpendicular on the tangent EP in P, and SP $=p$, $x=\frac{abb}{pp}$, then the ark AE will be the fluent of $\frac{\dot{x}\sqrt{ax}}{\sqrt{2ex-xx-bb}}$, or of $\frac{-a^2b^2\dot{p}}{pp\sqrt{2eap^2-a^2b^2-p^4}}$ that begins to be generated when $p=a$.

805. In order to represent the fluent of $\frac{\dot{z}}{\sqrt{z} \times \sqrt{b^2 \mp 2ez - zz}}$ or of $\frac{\dot{p}}{\sqrt{a^2b^2 \mp 2aep^2 - p^4}}$, we must have recourse to both the hyperbolic and elliptic arks. The rest remaining as in the last article, join AD, and let AF (*fig.* 308) perpendicular to AD meet DS produced in F, describe an ellipse ARb that has its focus in F, centre in S, and SA for the second semiaxis; upon SA take SQ equal to SP, let QR the ordinate at Q meet the ellipse in R, and $\frac{b}{a} \times$ AR + AE—EP shall be the fluent of $\frac{-bb\dot{p}}{\sqrt{aa-pp} \times \sqrt{bb+pp}}$

or

or (supposing $pp = az$, and $2ea = bb - aa$, as above) of $\frac{-bb\dot{z}}{2\sqrt{az} \times \sqrt{bb-2ez-zz}}$. For if RT the tangent of the ellipse at R meet SA in T, AR $= f$, SQ (= SP) $= p$, and SF $= k$, then ST $= \frac{aa}{p}$, QT $= \frac{aa-pp}{p}$, RT $= \frac{\sqrt{aa-pp}}{p} \times \sqrt{aa + \frac{k^2p^2}{a^2}}$, $\dot{f}$: $-\dot{p}$: : RT : QT, and $\dot{f} = \frac{-\dot{p}}{a} \times \frac{a^4+k^2p^2}{\sqrt{aa-pp^4}\sqrt{a^2+k^2p^4}} =$ (because $kb = aa$, by the supposition) $\frac{-a\dot{p}}{b} \times \frac{bb+pp}{\sqrt{aa-pp} \times \sqrt{bb+pp}}$. But the fluxion of EP—AE was found (art. 802) equal to $\frac{-pp\dot{p}}{\sqrt{aa-pp} \times \sqrt{bb+pp}}$. Therefore $\frac{b}{a} \times$ AR + AE—EP is the fluent of $\frac{-bb\dot{p}}{\sqrt{aa-pp} \times \sqrt{bb+pp}}$, or of $\frac{-bb\dot{z}}{2\sqrt{az} \times \sqrt{bb-2ez-zz}}$, that begins to be generated when p and z are equal to a; and that the fluent is finite which is generated while x decreases till it vanish, appears from art. 327.

806. The values of those fluents, as of $\frac{\dot{p}\sqrt{a^4+k^2p^2}}{a\sqrt{aa-pp}}$ (the fluxion of the elliptic ark BR), may be computed either by resolving $\sqrt{\frac{a^4+k^2p^2}{aa-pp}}$ into a series, multiplying each term by $\frac{\dot{p}}{a}$, and finding the fluents of the several products by art. 737, which will form a series of algebraic quantities. Or we may compare it with the fluent of $\frac{a\dot{p}}{\sqrt{aa-pp}}$ (the circular ark of the radius a and sine p), by resolving $\sqrt{a+k^2p^2}$ only into a series, by art. 793. Thus $\dot{f} = \frac{a\dot{p}}{\sqrt{aa-pp}} \times \overline{1 + \frac{k^2p^2}{2a^4} - \frac{k^4p^4}{8a^8} + \&c.}$; consequently the elliptic quadrant ARB is to the quadrant of the circle of the radius a as $1 + \frac{k^2}{4a^2} - \frac{3k^4}{64a^4} + \frac{5k^6}{256a^6} - \&c.$ to

to 1. The same elliptic quadrant is to the quadrant of the circle of a radius equal to SB the semi-transverse axis, supposing SB $= a$, as $1 - \frac{e^2}{4a^2} - \frac{3e^4}{64a^4} - \frac{5e^6}{256a^6} -$ &c. to 1.

807. Let $\frac{-bb\dot{p}}{\sqrt{aa-pp} \times \sqrt{bb+pp}} = \dot{Q}$, $\frac{-a\dot{p}}{\sqrt{aa-pp}} = \dot{A}$, then $\dot{Q} = \frac{-b\dot{p}}{\sqrt{aa-pp}} \times \overline{1+\frac{pp}{bb}}\Big|^{-\frac{1}{2}} = \frac{\dot{A}b}{a} \times \overline{1 - \frac{pp}{2bb} + \frac{3p^4}{8b^4} - \frac{5p^6}{16b^6} + \&c.}$ and, by art. 793, the fluent Q that is generated betwixt the terms, when $p = o$ and $p = a$, is (N being supposed to represent the ratio of the semi-circumference of a circle to its diameter) $Nb \times \overline{1 - \frac{a^2}{4b^2} + \frac{9a^4}{64b^4} - \frac{25a^6}{256b^6} + \&c.}$ where the numerical coefficients $1, \frac{1}{4}, \frac{9}{64}$, &c. are the squares of the several *unciæ* of a binomial raised to the power of the exponent $-\frac{1}{2}$. If we suppose $x = a - \frac{pp}{a}$, and $E = a + \frac{bb}{a}$, then $\dot{Q} = \frac{bb}{\sqrt{Ea}} \times \frac{\dot{x}}{2\sqrt{ax-xx}} \times \overline{1 - \frac{x}{E}}\Big|^{-\frac{1}{2}}$, or $\frac{bb}{a} \times \frac{\dot{x}}{2\sqrt{Ex-xx}} \times \overline{1 - \frac{x}{a}}\Big|^{-\frac{1}{2}}$.
Let A first denote the ark of a circle described upon the diameter a, whose versed sine is equal to x, and $\dot{A} = \frac{a\dot{x}}{2\sqrt{ax-xx}} = \frac{a}{2}\dot{x}x^{-\frac{1}{2}} \times \overline{a-x}|^{-\frac{1}{2}}$; whence $m - 1 = -\frac{1}{2}$, $m = \frac{1}{2}$, $n = 1$, $l = -\frac{1}{2}$, $e = a$, $f = 1$, $M = ln + n + m = 1$; and, by art. 792, when r is any integer positive number, the fluent of $\dot{A} \times \frac{x^r}{a^r}$, that is generated while x increases from o till it become equal to a, is $A \times \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} \times$ &c. these fractions being continued till their number be equal to r, and A being supposed now to represent the semi-circumference on the diameter a. Therefore, since $\dot{Q} = \frac{bb\dot{A}}{a\sqrt{aE}} \times \overline{1 + \frac{x}{2E} + \frac{3xx}{8EE} + \frac{5x^3}{16E^3} + \&c.}$ it follows, that $Q = \frac{Nbb}{\sqrt{aE}} \times \overline{1 + \frac{a}{4E} + \frac{9a^2}{64E^2} + \frac{25a^3}{256E^3} + \&c.}$

In like manner, $\dot{Q} = \frac{bb}{a} \times \frac{\dot{x}}{2\sqrt{Ex-xx}} \times \overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$. Whence Q may be compared with an ark of a circle upon the diameter E that has its versed sine equal to x.

808. Let $\dot{Q} = \frac{-p^2\dot{p}}{\sqrt{aa-pp} \times \sqrt{bb+pp}}$ and $\dot{A} = \frac{-a\dot{p}}{\sqrt{aa-pp}}$, then $\dot{Q} = \frac{\dot{A}p^2}{ab} \times \overline{1+\frac{pp}{aa}}|^{-\frac{1}{2}} = \frac{\dot{A}}{ab} \times p^2 - \frac{p^4}{2b^2} + \frac{3p^6}{8b^4} -$ &c. Therefore the fluent Q generated betwixt the terms when $p = o$ and $p = a$, is $\frac{Naa}{b} \times \overline{\frac{1}{2} - \frac{3a^2}{16b^2} + \frac{15a^4}{128b^4} - \frac{175a^6}{8 \times 256b^6} + \text{\&c.}}$ This fluent is the ultimate excess of the tangent EP above the hyperbolic ark AE, that is, the limit of this excess while the figure is produced, or (according to the usual manner of expression) the excess of the asymptote above the curve AE, when both are supposed to be infinitely produced. By supposing $aa-pp = ax$, $\dot{Q} = \frac{\dot{x}\sqrt{a} \times \sqrt{ax-xx}}{2x\sqrt{E-x}}$, and the same fluent will be found (by art. 792) equal to $\frac{Na}{2} \times \sqrt{\frac{a}{E}} + \frac{aA}{2\times 4E} + \frac{9aB}{4\times 6E} + \frac{25aC}{6\times 8E} + \frac{49aD}{8\times 10E} +$ &c. where A denotes in the usual manner the first term $\frac{Na}{2} \times \sqrt{\frac{a}{E}}$, B the second term $\frac{aA}{8E}$, C the third term, and so on.

809. It follows, from what was shown above, art. 792, 799, &c. that when r is any integer number, the fluent of $\frac{z^{\frac{rn}{4}-1}\dot{z}}{\sqrt{e+fz^n}}$ is assignable by the arks of conic sections; that is, by right lines, when r is equal to 4, or to any multiple of 4; by circular and parabolic arks (which may be reduced to logarithms) with right lines, when r is any other even number; by arks of an equilateral hyperbola with right lines, when r is any number of the series 3, 7, 11, 15, &c.; and by arks of the

same

same hyperbola and right lines, with arks of an ellipsis that has its excentricity equal to the second axis, when r is any of the numbers 1, 5, 9, 13, &c. For if we suppose $z^n = xx$, the proposed fluxion will be transformed into $\frac{2x^{\frac{r}{2}-1}\dot{x}}{n\sqrt{e+fxx}}$, when $r = 3$, $\frac{r}{2}-1 = \frac{1}{2}$, and the fluent is found by art. 799 or 800; but when $r = 1$, $\frac{r}{2}-1 = -\frac{1}{2}$, and the fluent is found by art. 802.

810. Let n (*fig.* 310) be any fraction whatsoever, and the fluent of $\frac{x^n\dot{x}}{\sqrt{xx-1}}$ or $\frac{z^{n-1}\dot{z}}{\sqrt{1-zz}}$ be required. For this end let AL be one of the figures constructed in art. 392, where the point S, and right line AE, were supposed to be given in position, SA was perpendicular to AE in A, M any point upon AE; and the ratio of the angle ASL to ASM, and that of the logarithm of the ray SL to the logarithm of SM, was always the same invariable ratio of n to 1; that is, ASL : ASM : : n : 1, and SL to SA as SM$^{\mp n}$ to SA$^{\mp n}$. Let SA $= 1$, SM $= x$, SL $= r$, and the ark AL $= s$; consequently $r = x^{\mp n}$, $\dot{r} = \mp nx^{\mp n-1}\dot{x}$, and (by art. 392) $\dot{s} : \mp\dot{r} : :$ SM : AM : : $x : \sqrt{xx-1}$, or $\dot{s} = \frac{nx^{\mp n}\dot{x}}{\sqrt{xx-1}}$. Therefore the fluent of $\frac{x^{\mp n}\dot{x}}{\sqrt{xx-1}}$ is $\frac{1}{n} \times s = \frac{1}{n} \times$ AL. If Am be perpendicular to SM in m, then S$m = \frac{1}{x}$; consequently, if we suppose S$m = z$, then the fluent of $\frac{z^{\pm n-1}\dot{z}}{\sqrt{1-zz}}$ will be equal to $\frac{1}{n} \times$ AL. By supposing $z = \sqrt{1-yy}$ and $n = 2-2k$, $\frac{\dot{y}}{\overline{1-yy}|^k}$ is transformed into $\frac{-z^{n-1}\dot{z}}{\sqrt{1-zz}}$, and the fluent is $\frac{1}{n} \times$ AL. By supposing $y^m = z^2$, $\frac{z^{\frac{nm}{2}-1}\dot{z}}{\sqrt{1-z^m}}$ is transformed into $\frac{2z^{n-1}\dot{z}}{m\sqrt{1-zz}}$, and the fluent is $\frac{2}{mn} \times$ AL.

These

These are the figures which we found to resolve the most simple cases of problems of various kinds in the first book, art. 436, 569, &c.

811. Let Al, Ap, AL, AP (*fig*. 311), &c. be such a series of figures as was described in art. 212, where each curve is supposed to be always defined by the intersections of the tangents of the preceding curve with the respective perpendiculars on those tangents drawn from the given point S. Let AL be a figure of the kind described in the last article; that is, let the angle ASL be to ASM, and *log*. SL to *log*. SM always in the same invariable ratio. Then Al, Ap, AP, and all the other figures in the series, shall be likewise of this kind. By art. 212, the angle ASl = ASL $\mp$ 2ASM, Sl $= x^{\mp n \mp 2}$, and the fluxion of Al to the fluxion of AL as the fluxion of Sl to the fluxion of SL, or as $\overline{n \mp 2} \times x^{\mp 2}$ to n; consequently the fluxion of AL is $\frac{n \mp 2}{n} \times \dot{x} x^{\mp 2}$; and the ark Al is assignable by s and algebraic quantities, by art. 792. The same is to be said of all the other arks in the series taken alternately, that is, of the 2d, 4th, 6th, &c. from AL. The other curves in the series Ap, AP, &c. are all assignable by AP and right lines; but the arks of any two figures that immediately succeed each other in the series, as of AP and AL, cannot be compared with each other by an algebraic equation. When ApS (*fig*. 311, *N*. 2) is supposed to be a semicircle upon the diameter SA, l coincides with A, and Al vanishes, AL and the subsequent arks in the series taken alternately (which have all a *cuspid* in S) are assignable by right lines; but the other arks in the series are measured by the circular ark Ap and right lines. When AL (*N*. 3) is supposed to coincide with the right line AM itself (or $n = 1$), P coincides with A, and AP vanishes, Ap is a common parabola that has its focus in S, Al and the arks in the series continued backwards, taken alternately from Al, admit of a perfect rectification; but the other arks in the same series are measured by parabolic arks and right lines. Of all the figures wherein the angle ASL is to the angle ASM and log. SL to log. SM in the same invariable ratio, there are none besides these that seem to

admit

admit of a perfect rectification, or an accurate mensuration by circular arks or logarithms. When AL is an equilateral hyperbola that has its centre in S (or $n=\frac{1}{2}$), the curves taken alternately from AL either way in the series, are measured by AL and right lines; but the other curves in the series are measured by AL with an elliptic ark (described above, art. 802) and right lines. By supposing $n=\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. other series of curves will be formed. And every series of such curves gives two distinct sorts of fluents, which cannot be compared with each other, or with those of any different series of this kind.

CHAP. IV.

Of the Area, when the Ordinate and Base are expressed by Fluents; of computing Fluents from the Sums of Progressions, or the Sums of Progressions from Fluents, and other Branches of this Method.

812. THE base being represented by z, and the ordinate of the figure by y, the fluxion of the area is $\dot{z}y$. If y and z be both assigned by quantities compounded in common algebraic terms from the powers of the same variable quantity x, the fluxion of the area will be expressed by such quantities multiplied by $\dot{x}$. Having insisted on the fluents of such expressions in the preceding chapters, we now proceed to enquire into the area or fluent when the ordinate is itself assigned by an area or fluent, or when the ordinate and base are both expressed by fluents: and in this case it will be sufficient if we can reduce the area of the figure to the fluents of the former kind; as to circular arks and logarithms, or to elliptic and hyperbolic arks, or, in general, to the fluents of expressions that involve one variable quantity only in algebraic terms with its fluxion. In this case we shall find that the total area (or that which insists upon a certain given base) may be sometimes measured by circular arks or logarithms, though it may not appear that

that in the same instances the part of the area can be assigned in this manner which stands upon any segment of the base that may be proposed. For example, let AD*a*, BE*b* (*fig.* 312), be concentric circles described from the same centre, CB being less than CA; let AG be the tangent at A, and T any point upon AG; join CT intersecting the circle BE*b* in V and *v*. Now, let the figure CHKR be constructed so that the base CR may be always equal to the logarithm of the ratio of CT + AT to CA, and the ordinate RK always equal to the logarithm of the ratio of T*v* to TV, the *modulus* being CA. Then the whole area CHKLLRC generated by the ordinate RK, while the point V describes the quadrant BVE, shall be equal to the rectangle contained by the quadrant AFD and the ark DF whose sine is CB; but it does not appear that the part of this area CHKR, that stands upon any given base CR, can be measured in this manner. The fluents of this kind are sometimes required in the resolution of useful problems, and the mensuration of the whole area is commonly what is most valuable. But before we treat of the area, when the ordinate and base are both expressed by fluents, some theorems are to be premised concerning the area, when the ordinate only is expressed in this manner.

813. Let A represent any area on the base x, suppose $A\dot{x} = \dot{K}$, $K\dot{x} = \dot{L}$, $L\dot{x} = \dot{M}$, $M\dot{x} = \dot{N}$, &c. where K represents the area when the ordinate is A, L the area when the ordinate is K, M the area when the ordinate is L, and so on. Let $x\dot{A} = \dot{B}$, $x\dot{B} = \dot{C}$, $x\dot{C} = \dot{D}$, $x\dot{D} = \dot{E}$, &c.; and suppose $B\dot{x} = \dot{i}$, $k\dot{x} = \dot{i}$, $l\dot{x} = \dot{m}$, $m\dot{x} = \dot{n}$, &c. Then shall $K = xA - B$, $2L = xK - k$, $3M = xL - l$, $4N = xM - m$, and so on. For since $A\dot{x} + x\dot{A} = \dot{K} + \dot{B}$, it follows, by finding the fluents (art. 738), that $Ax = K + B$, and $K = Ax - B$. Because $K\dot{x} + x\dot{K} - \dot{i} = \dot{L} + Ax\dot{x} - B\dot{x} = \dot{L} + \overline{Ax - B} \times \dot{x} = \dot{L} + K\dot{x} = 2\dot{L}$, by taking the fluents $xK - k = 2L$. In like manner, $L\dot{x} + x\dot{L} - \dot{i} = \dot{M} + Kx\dot{x} - k\dot{x} = \dot{M} + 2L\dot{x} = 3\dot{M}$, and $3M = xL - l$; $M\dot{x} + x\dot{M} - \dot{m} = \dot{N} + Lx\dot{x} - l\dot{x} = \dot{N} + 3M\dot{x} = 4\dot{N}$, and $4N = xM - m$, and so on.

814. In

814. In the same manner that $K = xA - B$, it is manifest that $k = xB - C$; consequently $2L = xK - k = x^2A - xB - xB + C = x^2A - 2xB + C$, and $2l = x^2B - 2xC + D$. Hence $6M =$ (by the last art.) $2xL - 2l = x \times \overline{x^2A - 2xB + C} - \overline{x^2B - 2xC + D} = x^3A - 3x^2B + 3xC - D$, and $6m = x^3B - 3x^2C + 3xD - E$; $24N = 6xM - 6m = x \times \overline{x^3A - 3x^2B + 3xC - D} - \overline{x^3B - 3x^2C + 3xD - E} = x^4A - 4x^3B + 6x^2C - 4xD + E$. And in this manner it is manifest, that if r denote the place of any fluent Z in the series

$$K,\ L,\ M,\ N,\ \&c.\ Z = \frac{x^rA - rx^{r-1}B + r \times \frac{r-1}{2} \times x^{r-2}C - \&c.}{1 \times 2 \times 3 \times 4 \times \ldots \times r}$$

which is the first part of *prop.* 11, *De Quadrat. Curvar.* When

$$x = a,\ \text{then}\ Z = \frac{a^rA - ra^{r-1}B + r \times \frac{r-1}{2}\, a^{r-2}C - \&c.}{1 \times 2 \times 3 \ldots \times r}$$

815. Let $\dot{z} = \overline{a-x}^r \times \dot{A} = a^r\dot{A} - ra^{r-1}x\dot{A} + r \times \frac{r-1}{2} \times a^{r-2}x^2\dot{A} - \&c. = a^r\dot{A} - ra^{r-1}\dot{B} + r \times \frac{r-1}{2} \times a^{r-2}\dot{C} - \&c.$; consequently $z = a^rA - ra^{r-1}B + r \times \frac{r-1}{2} \times a^{r-2}C - \&c.$ and when $x = a$, $Z = \frac{z}{1 \times 2 \times 3 \times \ldots \times r}$; which is the second part of the same proposition.

816. Let $x\dot{A} = \dot{P}$, $P\dot{A} = \dot{Q}$, $Q\dot{A} = \dot{R}$, $R\dot{A} = \dot{S}$, &c. and the fluent of $A^n\dot{x}$ will be equal to $xA^n - nA^{n-1}P + n \times \overline{n-1} \times A^{n-2}Q - n \times \overline{n-1} \times \overline{n-2} \times A^{n-3}R - \&c.$ For, by art. 738, the fluent of $A^n\dot{x}$ is $xA^n - F,\ nA^{n-1}x\dot{A}$ (where F is prefixed to denote the fluent of the expression that immediately follows) $= xA^n - F,\ nA^{n-1}\dot{P} = xA^n - nA^{n-1}P + F,\ n \times \overline{n-1} \times A^{n-2}\dot{Q} = xA^n - nA^{n-1}P + n \times \overline{n-1} \times A^{n-2}Q - F,\ n \times \overline{n-1} \times \overline{n-2} \times A^{n-3}\dot{Q}$, and so on.

817. For example, let $\dot{A} = \frac{\dot{x}}{\sqrt{1 \mp xx}}$, and K the fluent of $A\dot{x}$ will be $xA - B =$ (because $\dot{B} = x\dot{A} = \frac{x\dot{x}}{\sqrt{1 \mp xx}}$, and B

=

$= \mp \sqrt{1 \mp xx}) \, x\mathrm{A} \pm \sqrt{1 \mp xx}$; and, because the fluents B, C, D, &c. are expressed by circular arks or logarithms with algebraic quantities, according as A is itself a circular ark or logarithm, the same is to be said of the fluents K, L, M, N, &c. by art. 814. Let $\sqrt{1 \mp xx} = z$; then $\dot{\mathrm{A}} = \frac{\dot{x}}{z}$, $\dot{\mathrm{P}} = x\dot{\mathrm{A}} = \frac{x\dot{x}}{z} = \mp \dot{z}$, and $\mathrm{P} = \mp z$; $\dot{\mathrm{Q}} = \mathrm{P}\dot{\mathrm{A}} = \mp \dot{x}$, and $\mathrm{Q} = \mp x$; $\dot{\mathrm{R}} = \mathrm{Q}\dot{\mathrm{A}} = \mp \frac{x\dot{x}}{z} = \dot{z}$, and $\mathrm{R} = z$; $\dot{\mathrm{S}} = \mathrm{R}\dot{\mathrm{A}} = \dot{x}$, and $\mathrm{S} = x$. Therefore the fluent of $\mathrm{A}^n\dot{x}$ is $x\mathrm{A}^n \pm n\mathrm{A}^{n-1}z \mp n \times \overline{n-1} \times \mathrm{A}^{n-2}x - n \times \overline{n-1} \times \overline{n-2} \times \mathrm{A}^{n-3}z + n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \mathrm{A}^{n-4}x +$ &c.

818. Supposing $\dot{\mathrm{A}} = y\dot{x}$, $\dot{\mathrm{B}} = x\dot{\mathrm{A}} = yx\dot{x}$. If y can be expressed by x, $\dot{\mathrm{B}}$ may be expressed by a fluxion that involves an invariable quantity only *(viz. x)* with its fluxion; and if A and B can be reduced to algebraic quantities, or to circular arks or logarithms, by the preceding articles, the same is to be said of K, the fluent of $\mathrm{A}\dot{x}$; because $\mathrm{K} = x\mathrm{A} - \mathrm{B}$. It is obvious that if A and x be assignable by each other, $\mathrm{A}\dot{x}$ or $\dot{\mathrm{K}}$ may be easily expressed by a fluxion that involves one variable quantity only *(viz. x* or A) with its fluxion; and the fluent of $\mathrm{A}\dot{x}$ may, in many cases, be assigned in algebraic quantities, or compared with circular arks or logarithms, by the preceding articles. But besides these more obvious cases, there are others wherein the fluent of $\dot{x} \times \mathrm{F}, y\dot{x}$ (or of $\mathrm{A}\dot{x}$) can be reduced to such as have been considered above.

819. The base of a figure being represented by z, and the ordinate by y, let $\dot{z} = \dot{x}x^{rn-1} \times \overline{\mathrm{E} + \mathrm{F}x^n}|^{l-1}$, and $\dot{y} = \dot{x}x^{sn-1} \times \overline{e+fx^n}|^k$, and let $x = d$ when $\mathrm{E} + \mathrm{F}x^n = o$ (that is, let $d^n = \frac{-\mathrm{E}}{\mathrm{F}}$); then if $r + s + l + k = o$, the area of the figure (or the fluent of $\dot{z}y$) that is generated while x by flowing from o becomes equal to d, shall be equal to the simultaneous fluents

fluents of $x^{rn+sn-1}\dot{x} \times \overline{E+Fx^n}|^{l-1}$ and $x^{sn-1}\dot{x} \times \overline{e+fx^n}|^{k+l}$ multiplied by $\frac{1}{e^l d^{sn}}$; that is, let Q, G, and P, represent the several fluents of $\dot{x}x^{rn-1} \times \overline{E+Fx^n}|^{l-1} \times F.\ \dot{x}x^{sn-1} \times \overline{e+fx^n}|^k$, $\dot{x}x^{rn+sn-1} \times \overline{E+Fx^n}|^{l-1}$, and $\dot{x}x^{sn-1} \times \overline{e+fx^n}|^{k+l}$, that are generated while x by flowing from o becomes equal to d; and $Q = \frac{GP}{e^l d^{sn}}$. For, by the supposition, $\frac{\dot{y}}{e^k} = \dot{x}x^{sn-1} \times \overline{1+\frac{fx^n}{e}}|^k =$ (by the binomial theorem) $x^{sn-1}\dot{x} + \frac{kf}{e} \times x^{sn+n-1}\dot{x} + k \times \frac{k-1}{2} \times \frac{ff}{ee} \times x^{sn+2n-1}\dot{x}$ + &c. and (A) $\frac{sny}{e^k x^{sn}} = 1 + \frac{s}{s+1} \times \frac{kfx^n}{e} + \frac{s}{s+2} \times k \times \frac{k-1}{2} \times \frac{ffx^{2n}}{ee}$ + &c.; consequently $y\dot{x}$ is equal to the product of $\frac{e^k\dot{x}}{sn} \times x^{rn+sn-1} \times \overline{E+Fx^n}|^{l-1}$ multiplied by this series. Therefore, by art. 792, if in this series you substitute d for x, and multiply the terms respectively by 1, $\frac{r+s}{r+s+l}$, $\frac{r+s}{r+s+l} \times \frac{r+s+1}{r+s+l+1}$, &c. or (because $r+s = -l-k$, and $r+s+l = -k$) by 1, $\frac{l+k}{k}$, $\frac{l+k}{k} \times \frac{l+k-1}{k-1}$, &c. and suppose the series thence arising, viz. $1 + \frac{s}{s+1} \times \overline{k+l} \times \frac{fd^n}{e} + \frac{s}{s+2} \times \overline{k+l} \times \frac{k+l-1}{2} \times \frac{ffd^{2n}}{ee}$ + &c. $=$ L, we shall have $Q = \frac{e^k}{sn} \times$ GL. But by substituting in the equation A, by which the value of y was determined, P for y, $k+l$ for k, and d for x, it is manifest that $L = \frac{snP}{e^{k+l}d^{sn}}$; consequently $Q = \frac{GP}{e^l d^{sn}}$. This theorem is founded on art. 792, and is to be understood with similar limitations, particularly

larly with those described in art. 796. We have supposed $r+s+l+k=o$, or $s=-r-l-k$; but it is easy to see that if s be increased or diminished by any integer number, this theorem will be of use for discovering the fluent of $y\dot{z}$, when $x=d$, or for reducing it to common fluents, that is, to such as involve the powers of one variable quantity compounded together in common algebraic terms with the fluxion of that quantity. Suppose, for example, that $\dot{Q}=\dot{z}\times F,\dot{y}x^n$, and let $e+fx^n=R$, then $\dot{Q}=\frac{\dot{z}x^{sn}R^{k+1}}{nf\times\overline{s+k+1}}-\frac{sey\dot{z}}{f\times\overline{s+k+1}}$.

820. Let $x=D$ when $e+fx^n=o$; and, the values of $\dot{z}$ and $\dot{y}$ remaining the same as in the last article, let Y, Z, and q, be the respective fluents of $\dot{y}$, $\dot{z}$, and $y\dot{z}$, when $e+fx^n=o$. Let g and p be the simultaneous fluents of $\dot{x}x^{rn+sn-1}\times\overline{e+fx^n}^k$ and $\dot{x}x^{rn-1}\times\overline{E+Fx^n}^{l+k}$. Then if $r+s+l+k=o$, as formerly, $q=YZ-\frac{gp}{E^{k+1}D^{rn}}$. This theorem follows from the last, because $F,y\dot{z}=yz-F,z\dot{y}$.

821. Let $\dot{z}=-\dot{x}x^{n-m-2}\times\overline{Ex^n+F}^{l-1}$, $\dot{y}=-\dot{x}x^m\times\overline{ex^n+f}^k$, $x=d$ when $E+Fx^n=o$, and the area or fluent of $y\dot{z}$ which is generated while x flows till from being equal to d it becomes infinitely great (or the limit to which this area approximates while x increases continually), is the product of the fluents of $-\dot{x}x^{n+nk-1}\times\overline{Ex^n+F}^{l-1}$ and $-\dot{x}x^{m-nl}\times\overline{ex^n+f}^{k+l}$ multiplied by each other and by $\frac{1}{e^l d^{m+nk+1}}$; as will appear by substituting x^{-1} for x in art. 819, and supposing $m=rn+ln-1$. The fluent of $y\dot{z}$ when $e+fx^n=o$ is the excess of the product of the corresponding values of y and z above the product of the simultaneous fluents of $-\dot{x}x^{nl-1}\times\overline{ex^n+f}^k$, and $-\dot{x}x^{-m-nk-2}\times\overline{Ex^n+F}^{l+k}$ multiplied by each other and by $\frac{D^{m+1-nl}}{E^{k+1}}$, by the last article.

822. From

822. From these theorems tables might be computed of fluents of this kind that may be reduced to circular arks and logarithms; but we shall only give a few examples of their use. Suppose that it is required to find this area or fluent of $y\dot{z}$, when, m being any positive number, $\dot{z} = \frac{\dot{x}}{x^m \sqrt{bb - xx}}$ and $\dot{y} = \frac{x^{m-1}\dot{x}}{aa \mp xx}$;

that is, let it be required to find the fluent of $\frac{\dot{x}}{x^m \sqrt{bb-xx}} \times F$, $\frac{x^{m-1}\dot{x}}{aa \mp xx}$ when $x = b$. In this case, by comparing the exponents with those in art. 819, $n=2$, $r = \frac{1-m}{2}$, $l = \frac{1}{2}$, $s = \frac{m}{2}$, and $k = -1$, so that $r + l + s + k = \frac{1-m+1+m}{2} - 1 = 0$, as the theorem requires. Because in this case $\dot{G} = \frac{\dot{x}}{\sqrt{bb-xx}}$, and $\dot{P} = \frac{x^{m-1}\dot{x}}{\sqrt{aa \mp xx}}$; it follows, that if N denote the ratio of the circumference of a circle to its diameter, the fluent required is $\frac{N}{2ab^m} \times F. \frac{x^{m-1}\dot{x}}{\sqrt{aa \mp xx}}$, if b be substituted for x in the value of this last fluent after it is determined. Thus if $z = \log. \frac{bb - b\sqrt{bb-xx}}{x}$ and $y = \log. a \times \sqrt{\frac{a+x}{a-x}}$, and H represent the ark described with the radius a that has its sine equal to b, then the area required will be equal to $\frac{Nb}{2} \times H$; whence the proposition that was advanced in art. 812 follows: for in this example $\dot{z} = \frac{bb\dot{x}}{x\sqrt{bb-xx}}$, $\dot{y} = \frac{aa\dot{x}}{aa-xx}$, $m = 1$ and $P = F. \frac{\dot{x}}{\sqrt{aa-xx}}$. If, the same value of z remaining, we suppose y to be always equal to the ark described with the radius a that has its tangent equal to x, or $\dot{y} = \frac{aa\dot{x}}{aa+xx}$, then $\dot{P} = \frac{\dot{x}}{\sqrt{aa+xx}}$; and in this example

the fluent required is $\frac{Nba}{2} \times \log. \frac{\sqrt{aa+bb}+b}{a}$; so that the fluent, which in the former case was the product of two circular arks, is now the product of a circular ark by a logarithm. If m be any integer number, the fluent required may be measured by the areas of conic sections, and if m be equal to any of the fractions $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, &c. it may be measured by their arks.

823. It is obvious that in other cases the fluents may be computed from the theorem besides those wherein $r+s+l+k=0$. Suppose, for example, $\dot{z} = \frac{-\dot{x}}{x^3\sqrt{bb-xx}}$ and $\dot{y} = \frac{e\dot{x}}{e+fxx}$. Because $\dot{y} = \dot{x} - \frac{fx^2\dot{x}}{e+fxx}$, $\dot{z}y = \frac{\dot{x}}{xx\sqrt{bb-xx}} - \frac{\dot{x}}{x^3\sqrt{bb-xx}} \times F, \frac{-fx^2\dot{x}}{e+fxx}$. The fluent of the former part is assignable in algebraic terms, and the fluent of the latter (by the theorem in art. 819) by circular arks and right lines.

824. In like manner it follows, from art. 822, that if $\dot{z} = \frac{-\dot{x}}{x^h\sqrt{xx-bb}}$ and $\dot{y} = \frac{-x^h\dot{x}}{xx\mp aa}$, then the area, or fluent of $y\dot{z}$ that is generated while x flows till from being equal to b it become infinite, is $\frac{N}{2b^h} \times F, \frac{x^{h-1}\dot{x}}{\sqrt{xx\mp aa}}$.

825. The theorems in art. 819, &c. are chiefly of use for reducing fluents to circular arks or logarithms, or to others of a more simple form (and consequently for rendering our solutions of problems more simple and elegant than when we have immediately recourse to an infinite series), when neither y nor z can be expressed by x in algebraic terms. But they may be of some use, likewise, for finding the fluent of $y\dot{x}$ when y is assigned by x. Thus, to find the fluent of $\frac{aa\dot{x}}{aa+xx \times \sqrt{bb-xx}}$ when $x=b$, suppose $\dot{z} = \frac{\dot{x}}{\sqrt{bb-xx}}$, $y = \frac{aa}{aa+xx}$, and conse-

quently

quently $\dot{y} = \frac{-2aax\dot{x}}{\overline{aa+xx}|^2}$. By comparing these values of $\dot{x}$ and $\dot{y}$ with their general values in art. 819, $n = 2$, $r = \frac{1}{2}$, $l = \frac{1}{2}$, $s = 1$, $k = -2$, and $r + l + s + k = o$, as the theorem requires, G the fluent of $\frac{x^A\dot{x}}{\sqrt{bb-xx}}$ is $\frac{Nbb}{4}$, and P the fluent of $\frac{-x\dot{x}}{\overline{aa+xx}|^{\frac{3}{2}}}$ is $\frac{1}{\sqrt{aa+xx}}$; whence, by the theorem in art. 819, the fluent required is $\frac{Na}{2\sqrt{aa+bb}}$. Other examples might be given, if we were not obliged to hasten towards a conclusion, this Treatise having already grown to a far greater bulk than was at first intended.

826. If we assume an equation as $\overline{x+Ay}|^m \times \overline{x+By}|^n = E$, where m, n, A, B, E, are supposed invariable, then (by art. 728) $\frac{m\dot{x}+mA\dot{y}}{x+Ay} + \frac{n\dot{x}+nB\dot{y}}{x+By} = o$, and $\overline{m+n} \times x\dot{x} + \overline{nA+mB} \times y\dot{x} + \overline{mA+nB} \times x\dot{y} + \overline{m+n} \times AB y\dot{y} = o$. If we had assumed $\overline{x+Ay}|^m \times y^n = E$, then $my\dot{x} + nx\dot{y} + \overline{m+n} \times Ay\dot{y} = o$, where the term $x\dot{x}$ is wanting. When a fluxional equation is proposed that can be reduced to a form of this kind, then, by comparing its coefficients with those of the equation of the same form, the values of m, n, A, &c. may be determined, and the equation for the fluents discovered; as is shown more fully, *Comment. Petropol. tom.* 1, &c.

827. When an area or fluent is reduced to a series by the methods described in art. 745, &c. the series in some cases converges at so slow a rate as to be of little use for finding the area. Suppose FMf (*fig.* 313) to be an equilateral hyperbola that has its centre in O and Oa for one of its asymptotes; let OA $= 1$, AP$= x$, PM $= y$, and $y = \frac{1}{1+x} = 1 - x + x^2 - x^3 +$ &c. whence the area APMF $=$ F, $y\dot{x} = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} +$ &c; and

if AB = 1, the area ABEF $= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$ &c. $= \frac{1}{2} + \frac{1}{12} + \frac{1}{30} +$ &c. which is the series mentioned for the quadrature of the hyperbola (or for finding the hyperbolic logarithm of 2) in art. 361. But this series converges so slowly, that the sum of the first 1000 terms* of it is found deficient from the true value of the area in the fifth decimal; and other examples similar to this might be brought, wherein the area may be more easily computed from the inscribed polygons than from the series. Some further artifice is therefore necessary in order to compute the area in such cases, instances of which were described in art. 361, and others are to be met with in several authors, particularly those who have treated of the computation of logarithms and mensuration of the circle. The following theorems, derived from the method of fluxions, may be of use for this purpose; and serve for the resolution of many problems that are usually referred to what is called Sir *Isaac Newton's differential method.*

828. Suppose the base AP $= z$ (*fig.* 314), the ordinate PM $= y$, and, the base being supposed to flow uniformly, let $\dot{z} = 1$. Let the first ordinate AF be represented by a, AB $= 1$, and the area ABEF $=$ A. As A is the area generated by the ordinate y, so let B, C, D, E, F, &c. represent the areas upon the same base AB generated by the respective ordinates $\dot{y}, \ddot{y}, \dddot{y}, \ddddot{y}$, &c. Then AF $= a =$ A $- \frac{B}{2} + \frac{C}{12} - \frac{E}{720} + \frac{G}{30240} -$ &c. For, by art. 752, A $= a + \frac{\dot{a}}{2} + \frac{\ddot{a}}{6} + \frac{\dddot{a}}{24} + \frac{\ddddot{a}}{120} +$ &c. whence we have the equation (Q) $a = A - \frac{\dot{a}}{2} - \frac{\ddot{a}}{6} - \frac{\dddot{a}}{24} - \frac{\ddddot{a}}{120} -$ &c. In like manner, $\dot{a} = B - \frac{\ddot{a}}{2} - \frac{\dddot{a}}{6} - \frac{\ddddot{a}}{24} -$ &c. $\ddot{a} = C - \frac{\dddot{a}}{2} - \frac{\ddddot{a}}{6} -$ &c. $\dddot{a} = D - \frac{\ddddot{a}}{2} -$ &c. $\ddddot{a} = E -$ &c. by which lat-

* Stirling *De Summat. Serierum, p.* 28.

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ter equations, if we exterminate $\dot{a}$, $\ddot{a}$, $\dddot{a}$, $\ddddot{a}$, &c. from the value of a in the equation Q, we shall find that $a = A - \frac{B}{2} + \frac{C}{12} - \frac{E}{720} +$ &c. The coefficients are continued thus: let k, l, m, n, &c. denote the respective coefficients of $\dot{a}$, $\ddot{a}$, $\dddot{a}$, &c. in the equation Q; that is, let $k = \frac{1}{2}$, $l = \frac{1}{6}$, $m = \frac{1}{24}$, $n = \frac{1}{120}$, &c.; suppose $K = k = \frac{1}{2}$, $L = kK - l = \frac{1}{12}$, $M = kL - lK + m = o$, $N = kM - lL + mK - n = -\frac{1}{720}$, and so on; then $a = A - KB + LC - MD + NE -$ &c. where the coefficients of the alternate areas D, F, H, &c. vanish.

829. As A is the fluent of $y\dot{z}$, so B is the fluent of $\dot{y}\dot{z}$, C of $\ddot{y}\dot{z}$, E of $\ddddot{y}\dot{z}$, &c. Therefore, since $\dot{z} = 1$, and these areas are generated while the ordinate PM moves from AF to BE, the area B will be expressed by the excess of the last ordinate BE above the first AF, C by the difference of the fluxions of the ordinates (having due regard to the signs of these fluxions), E by the difference of their third fluxions, and the other areas G, I, &c. by the respective differences of their fluxions of the corresponding higher orders. Therefore if α represent BE—AF the difference of the ordinates, and β, δ, ζ, &c. the differences of their fluxions of the first, third, fifth orders, &c. then $a = A - \frac{\alpha}{2} + \frac{\beta}{12} - \frac{\delta}{720} + \frac{\zeta}{30240} +$ &c.

830. Supposing now the base Aa to be divided into the equal successive parts AB, BC, CD, &c. and each part equal to unit, let the sum of the equidistant ordinates AF, BE, CK, &c. exclusive of the last ordinate af, be represented by S, the total area $AFfa$ upon the base Aa by A, the excess of af above AF by α, the respective excesses of their first, third, fifth fluxions, &c. by β, δ, ζ, &c. the fluxion of the base being supposed equal to 1, then it follows, from the last article, that $S = A - \frac{\alpha}{2} + \frac{\beta}{12}$

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$-\frac{\delta}{720}+\frac{\zeta}{30240}+$ &c. and $A=S+\frac{\alpha}{2}-\frac{\beta}{12}+\frac{\delta}{720}-\frac{\zeta}{30240}+$ &c. which give two of the theorems mentioned in art. 352 and 353, where we had hyperbolic figures chiefly in view. This proposition more generally expressed, without supposing z or $\dot{z}$ equal to unit, is that $S=\frac{A}{\dot{z}}-\frac{\alpha}{2}+\frac{z\beta}{12\dot{z}}-\frac{z^3\delta}{720\dot{z}^3}+\frac{z^5\zeta}{30240\dot{z}^5}-\frac{z^7\theta}{1209600\dot{z}^7}+$ &c.

831. The ordinate AF being still represented by a, let AR and Ar be taken on opposite sides of the point A equal to each other, RV and rv the ordinates at R and r terminate the area RVvr; let y represent any ordinate as PM of the figure, and, the base being supposed to flow uniformly, let A, C, E, &c. represent the areas upon the base Rr that are generated by the respective ordinates $y, \ddot{y}, \ddddot{y}$, &c.; then, supposing AR$=z$, the middle ordinate AF ($=a$) $=\frac{A}{2z}-\frac{zC}{12\dot{z}^2}+\frac{7z^3E}{720\dot{z}^4}-\frac{31z^5G}{30240\dot{z}^6}$ $+$ &c.; for, by art. 752, $\frac{RrvV}{2z}=\frac{A}{2z}=a+\frac{z^2\ddot{a}}{2\times3\dot{z}^2}+\frac{z^4\ddddot{a}}{2\times3\times4\times5\dot{z}^4}+$ &c. or $a=\frac{A}{2z}-\frac{z^2\ddot{a}}{6\dot{z}^2}-\frac{z^4\ddddot{a}}{120\dot{z}^4}-$ &c. In like manner, $\ddot{a}=\frac{C}{2z}-\frac{z^2\ddddot{a}}{6\dot{z}^2}-$ &c. $\ddddot{a}=\frac{E}{2z}-$ &c.; whence, by exterminating $\ddot{a}, \ddddot{a}$, &c. from the value of a, the theorem will appear. The coefficients of C, E, G, &c. are continued thus: let the several coefficients of $\ddot{a}, \ddddot{a}$, &c. in the value of $\frac{A}{2z}$ (derived from art. 752) be represented by k, l, m, n, &c. that is, let $k=\frac{z^2}{2\times3\dot{z}^2}$, $l=\frac{z^2k}{4\times5\dot{z}^2}$, $m=\frac{z^2l}{6\times7\dot{z}^2}$, $n=\frac{z^2m}{8\times9\dot{z}^2}$, &c.; then let $K=\frac{k}{2z}=\frac{z}{12\dot{z}^2}$, $L=kK-\frac{l}{2z}$

$\frac{l}{2\dot{z}}$, $M = kL - lK + \frac{m}{2\dot{z}}$, $N = kM - lL + mK - \frac{n}{2\dot{z}}$, &c. And the values of the coefficients K, L, M, N, &c. being thus computed, then $a = \frac{A}{2\dot{z}} - KC + LE - MG + NI -$ &c. Because the areas C, E, &c. are the respective fluents of $\ddot{y}\dot{z}$, $\ddddot{y}\dot{z}$, &c. if the respective differences of the first, third, fifth, seventh, and higher alternate fluxions of the ordinates rv and RV, be expressed by β, δ, ζ, θ, &c. then $a = \frac{A}{2z} - \frac{z\beta}{12\dot{z}} + \frac{7z^3\delta}{720\dot{z}^3} - \frac{31z^5\zeta}{30240\dot{z}^5} + \frac{127z^7\theta}{1209600\dot{z}^7} -$ &c.

832. From this it follows, that if AF, BE, CK (*fig.* 315) &c. be a series of equidistant ordinates upon the base Aa, of which AF is the first and af the last; AB their common distance be equal to $2z$; AR be taken backwards from A equal to z or $\frac{1}{2}$ AB, and ar be taken forwards from a also equal to $\frac{1}{2}$ AB; the ordinates RV and rv terminate the area RVvr; and this area being represented by A, the differences by which the first, third, fifth, seventh, and higher alternate fluxions of rv exceed the same fluxions of RV, be expressed by β, δ, ζ, θ, &c. and the sum of the ordinates AF, BE, CK, &c. (including af) by S, then

$$S = \frac{A}{2z} - \frac{z\beta}{12\dot{z}} + \frac{7z^3\delta}{720\dot{z}^3} - \frac{31z^5\zeta}{30240\dot{z}^5} + \frac{127z^7\theta}{1209600\dot{z}^7} - \frac{511z^9\kappa}{47900160\dot{z}^9} + \text{\&c.}$$

$$A = 2zS + \frac{z^2\beta}{6\dot{z}} + \frac{7z^4\delta}{360\dot{z}^3} - \frac{31z^6\zeta}{15120\dot{z}^5} + \frac{127z^8\theta}{604800\dot{z}^7} - \frac{511z^{10}\kappa}{23950080\dot{z}^9} + \text{\&c.}$$

If we suppose AB $= 1$, and $\dot{z} = 1$, then $z = \frac{1}{2}$ and $S = A - \frac{\beta}{24} + \frac{7\delta}{5760} - \frac{31\zeta}{967680} + \frac{127\theta}{154828800} -$ &c. and $A = S + \frac{\beta}{24} - \frac{7\delta}{5760} + \frac{31\theta}{967680} -$ &c. which are the two other theorems mentioned in art. 352 and 353, only, in order to include the term af, ar is here taken forwards from a, whereas af was there excluded, and ar taken the contrary way.

 833. The

833. The use of these theorems will best appear by examples. First, let $m, m+e, m+2e, m+3e, \ldots n$, be a series of numbers in arithmetical progression, where m denotes the first term, e the common difference, and n the last term; and r being any number positive or negative (-1 excepted) S the sum of the powers of these numbers of the exponent r, that is, $m^r + \overline{m+e}^r + \overline{m+2e}^r + \overline{m+3e}^r + \ldots + n^r = \frac{1}{\overline{r+1}\times e} \times \overline{n^r - m^r} + \frac{n^r+m^r}{2} + \frac{re}{12} \times \overline{n^{r-1} - m^{r-1}} - \frac{r.r-1.r-2.e^3}{720} \times \overline{n^{r-3} - m^{r-3}} +$ &c. For, supposing $OP = x$, $PM = y$, let (*fig.* 314, N. 1 & 2) FM*f* be the parabola or hyperbola whose equation is $y = x^r$, $OA = m$, $Oa = n$; consequently $Af = m^r$, $af = n^r$, F. $\dot{y} =$ F. $x^m\dot{x} = \frac{x^{m+1}}{m+1}$ and the area $AFfa = A = \frac{n^{r+1} - m^{r+1}}{m+1}$; $af - AF = \alpha = n^r - m^r$; $\dot{y} = rx^{r-1}\dot{x}$, and, supposing $\dot{x} = \dot{z} = 1$, the difference of the fluxions of af and AF is $rn^{r-1} - rm^{r-1} = \beta$; $\dddot{y} = r \times \overline{r-1} \times \overline{r-2} \times x^{r-3}\dot{x}$, and $\delta = r \times \overline{r-1} \times \overline{r-2} \times \overline{n^{r-3} - m^{r-3}}$. In like manner, ζ, θ, &c. are computed, and it follows, from art. 830, that $S - n^r = \frac{n^{r+1} - m^{r+1}}{\overline{r+1}\times e} - \frac{n^r - m^r}{2} + \frac{re}{12} \times \overline{n^{r-1} - m^{r-1}} -$ &c. therefore $S = \frac{n^{r+1} - m^{r+1}}{\overline{r+1}\times e} + \frac{n^r + m^r}{2} + \frac{re}{12} \times \overline{n^{r-1} - m^{r-1}} -$ &c. By supposing $e = 1$ and $m = o$, it follows, that the sum of the powers of the numbers $0, 1, 2, 3, 4, \ldots n$ of any integer and positive exponent r is $\frac{n^{r+1}}{r+1} + \frac{n^r}{2} + \frac{rn^{r-1}}{12} - \frac{r \times \overline{r-1} \times \overline{r-2} \times n^{r-3}}{720} +$ &c. this series being continued to as many terms as there are units in $2 + \frac{r-1}{2}$ only, when r is an odd number: because when $r = 1$, the fluxions of AF and af are equal, and

$\beta = a;$

$\beta = o$; when $r = 3$, $\delta = o$; when $r = 5$, $\zeta = o$, &c. Thus if $r = 1$, $S = \frac{n^2}{2} + \frac{n}{2}$; if $r = 2$, $S = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$; and if $r = 3$, $S = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$. This is the theorem given by Mr. *James Bernouilli, Ars. Conjectandi, p.* 97. When r is a fraction or negative number, the sum of the powers of the same numbers (by supposing $m = 1$) is $\frac{n^{r+1} - 1}{r+1} + \frac{n^{r+1}}{2} + \frac{rn^{r-1} - r}{12} - \frac{r \times \overline{r-1} \times \overline{r-2}}{720} \times \overline{n^{r-3} - 1} + \&c.$

834. The sum S (*fig.* 315) of the same powers of $m + e$, $m + 3e$, $m + 5e$, $m + 7e$, ... $n - e$, where $2e$ is the common difference of the terms, computed by the theorem in art. 832, by supposing OR $= m$, RA $= e = ar$, $Or = n$ (and computing the area RVvr with the differences of the first, third, and higher alternate fluxions of rv and RV), is $\frac{1}{\overline{r+1} \times 2e} \times \overline{n^{r+1} - m^{r+1}} - \frac{re}{12} \times \overline{n^{r-1} - m^{r-1}} + \frac{7r \times \overline{r-1} \times \overline{r-2} \times e^3}{720} \times \overline{n^{r-3} - m^{r-3}} - \frac{31r \times \overline{r-1} \times \overline{r-2} \times \overline{r-3} \times \overline{r-4}}{30240} \times e^5 \times \overline{n^{r-5} - m^{r-5}} + \&c.$

By supposing $m = e = \frac{1}{2}$, the numbers are 1, 2, 3, 4, 5, ... $n - \frac{1}{2}$, and $S = \frac{n^{r+1}}{r+1} - \frac{rn^{r-1}}{24} + \frac{7r \times \overline{r-1} \times \overline{r-2} \times n^{r-3}}{5760} - \frac{31r \times \overline{r-1} \times \overline{r-2} \times \overline{r-3} \times \overline{r-4}}{967680} \times n^{r-5} + \&c. - \frac{1}{\overline{r+1} \times 2^{r+1}} + \frac{r}{24 \times 2^{r-1}} - \frac{7r \times \overline{r-1} \times \overline{r-2}}{5760 \times 2^{r-3}} + \&c.$

835. When r is negative, let $r = -s$; and if s be greater than unit, then, by what we have shown in article 833, the sum of the progression $\frac{1}{m^s} + \frac{1}{\overline{m+e}^s} + \frac{1}{\overline{m+2e}^s} + \frac{1}{\overline{m+3e}^s} + \&c.$ (by supposing $\frac{1}{n^s} = 0$) $= \frac{1}{\overline{s-1} \times em^{s-1}} + \frac{1}{2m^s} +$

$+\frac{se}{12m^{s+1}} - \frac{s\times\overline{s+1}\times\overline{s+2}\times e^3}{720m^{s+3}} + \frac{s\times\overline{s+1}\times\overline{s+2}\times\overline{s+3}\times\overline{s+4}}{30240m^{s+5}}$ $\times e^5$ — &c. This series was deduced from different principles by Mr. *De Moivre*. In like manner it follows, from the last article, that $\frac{1}{\overline{m+e}^s} + \frac{1}{\overline{m+3e}^s} + \frac{1}{\overline{m+5e}^s} + \frac{1}{\overline{m+7e}^s} +$ &c. $= \frac{1}{\overline{s-1}\times 2em^{s-1}} - \frac{se}{12m^{s+1}} + \frac{7s\times\overline{s+1}\times\overline{s+2}\times e^3}{720m^{s+3}} -$ &c. For example, if $s = 2$ and $e = \frac{1}{2}$, then $S = \frac{1}{m} - \frac{1}{12m^3} + \frac{7}{240m^5} - \frac{31}{1344m^7} +$ &c. To compute the sum of the progression $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} +$ &c. find the sum of the terms at the beginning of the series as far as $\frac{1}{\overline{m+\frac{1}{2}}^2}$ exclusively, then compute the sum of the subsequent terms by this theorem. Thus if we add the three first only $1 + \frac{1}{4} + \frac{1}{9} =$ 1.36111 &c. suppose $m + \frac{1}{2} = 4$, or $m = \frac{7}{2}$, and the sum of the following terms will be $\frac{2}{7} \times \overline{1 - \frac{1}{3\times 49} + \frac{1}{15\times 343} - \text{\&c.}}$ three terms of which series only collected and added to the former number 1.36111 &c. give 1.64493 &c. for the sum of the series required, true to the fifth decimal. If $s = \frac{1}{2}$, and $e = \frac{1}{2}$, then $S = \frac{1}{\sqrt{m}} \times \overline{2 - \frac{1}{16mm} + \frac{49}{3072m^4} - \frac{31\times 11}{32\times 1024m^6}} +$ &c.

836. These theorems may serve likewise, in many cases, for computing the area when the series that arises in the common method (described above, art. 745, &c.) converges at too slow a rate. For example, let V*mv* be a common hyperbola, O the centre, OR $= m$, O*r* $= n$, and R*r* be divided into any even number of equal parts of which RA is the first and *ar* the last; let RA $= e$, and S denote the sum of the ordinates AF, BE, . . . *af*, that insist upon the base at the distance $2e$ from each other.

other. Then the area $RVvr = 2eS + \frac{2e^2}{12} \times \overline{\frac{1}{mm} - \frac{1}{nn}} - \frac{2\times2\times3\times7e^4}{720} \times \overline{\frac{1}{m^4} - \frac{1}{n^4}} + \frac{2\times2\times3\times4\times5\times31e^6}{30240} \times \overline{\frac{1}{m^6} - \frac{1}{n^6}}$ — &c. because in this case $y = \frac{1}{x}$, $\dot{y} = -\frac{\dot{x}}{x^2}$, $\dddot{y} = -\frac{6\dot{x}^3}{x^4}$, &c. Hence, if $OR = m = 1$, $Or = n = 2$, and $RA = e = \frac{1}{8}$, then $S = \frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}$, and $2eS = \frac{2}{9} + \frac{2}{11} + \frac{2}{13} + \frac{2}{15}$; $\frac{2e^2}{12} \times \overline{\frac{1}{mm} - \frac{1}{nn}} = \frac{1}{512}$; $\frac{2\times2\times3\times7e^4}{720} \times \overline{\frac{1}{m^4} - \frac{1}{n^4}} = \frac{7}{64\times64\times64}$; and by so few sub-divisions of the base Rr and terms of the series, the area $RrvV$, or the hyperbolic logarithm of 2, is 0.693146 &c, which differs from the truth by an unit only in the sixth decimal.

837. The logarithm of m being given, the logarithm of $m + z$ is assigned by this theorem, log. $\overline{m+z}$ — log. $m = \frac{2z}{2m+z}$ multiplied by the series $1 - \frac{z}{12} \times \overline{\frac{1}{m+z} - \frac{1}{m}} + \frac{z^3}{360} \times \overline{\frac{1}{\overline{m+z}^3} - \frac{1}{m^3}} - \frac{z^5}{1260} \times \overline{\frac{1}{\overline{m+z}^5} - \frac{1}{m^5}}$ + &c. For let (*fig.* 316) $EA = m$, $Aa = z$, FMf the logarithmic curve having its asymptote EZ perpendicular to EA, $EP = x$, $PM = y$, the *modulus* or ordinate $Ee = 1$; then by the nature of the figure (art. 178), $\dot{y} = \frac{\dot{x}}{x}$, or $x\dot{y} = \dot{x}$; consequently, MN being perpendicular to the asymptote in N, the area $Ee\ FMN = x - 1$, $eMP = EP \times PM - Ee\ FMN = x \times \log. x - x + 1$, and the area $AFfa = \overline{m+z} \times \log. \overline{m+z} - m \times \log. m - z = A$. And because $a = af - AF = \log. \overline{m+z} - \log. m$ (supposing $\dot{z} = 1$), $AF = a = \log. m =$ (art. 829) $\frac{A}{z} - \frac{a}{2} + \frac{z\beta}{12} - \frac{z^3\delta}{720} + \frac{z^5\zeta}{30240}$ - &c. $= \overline{\frac{m}{z} + \frac{1}{2}} \times \log. \overline{m+z} - \overline{\frac{m}{z} - \frac{1}{2}} \times \log. m - 1 + \frac{z\beta}{12} - \frac{z^3\delta}{720} + \frac{z^5\zeta}{30240}$ — &c. Therefore $\overline{\frac{m}{z} + \frac{1}{2}} \times \overline{\log. \overline{m+z} - \log. m} = 1 -$

$\frac{z\beta}{12}$

$\frac{z\beta}{12} + \frac{z^3\delta}{720} - \frac{z^5\zeta}{30240} + \&c. =$ (because $\dot{y} = \frac{\dot{x}}{x}$ and $\beta = \frac{1}{m+z} - \frac{1}{m}$, $\ddot{y} = \frac{2\dot{x}^3}{x^3}$ and $\delta = \frac{2}{\overline{m+z}^3} - \frac{2}{m^3}$, &c.) $1 - \frac{z}{12} \times \overline{\frac{1}{m+z} - \frac{1}{m}} + \frac{z^3}{360} \times \overline{\frac{1}{\overline{m+z}^3} - \frac{1}{m^3}}, - \frac{z^5}{1260} \times \overline{\frac{1}{\overline{m+z}^5} - \frac{1}{m^5}}$. Hence, if we suppose $m = 1$, and $z = 1$, because log. $1 = o$, it follows, that $\frac{3}{2} \times \log. 2 = 1 + \frac{1}{12 \times 2} - \frac{7}{360 \times 8} + \frac{31}{1260 \times 32} - \&c.$ And by supposing $z = \frac{1}{2}$, $\frac{5}{2} \times \log. \frac{3}{2} = 1 + \frac{1}{72} - \frac{19}{360 \times 8 \times 27} + \frac{311}{1260 \times 32 \times 343} - \&c.$ By a similar computation, it appears that if z denote the excess of the logarithm of $a + d$ above the logarithm of a, or measure the ratio of $a + d$ to a, then d the difference of the numbers may be found by dividing az by the series $1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30240} - \&c.$ Other theorems of this kind may be derived from art. 832.

838. Let it be required to find the sum of the logarithms of a series of numbers $m + e$, $m + 3e$, $m + 5e$, $m + 7e$ .... $n - e$, in arithmetical progression, where $m + e$ denotes the least term, $n - e$ the greatest, and $2e$ the common difference of the terms; or, to find the logarithm of the product $\overline{m+e} \times \overline{m+3e} \times \overline{m+5e} \times \overline{m+7e} \times \ldots \times \overline{n-e}$, when all these numbers are supposed to be multiplied continually by one another. For this end, the figure being the same as in the last article, let EA be now equal to $m + e$, Ea $= n - e$, take AR from A towards E equal to e, and ar the contrary way equal to AR, and still suppose the fluxion of the base equal to 1; then ER $= m$, E$r = n$, the area RVvr $= n \times \log. n - m \times \log. m - n + m = $ A, the difference of the fluxions of the ordinates rv and RV, is $\frac{1}{n} - \frac{1}{m} = \beta$, $\delta = \frac{2}{n^3} - \frac{2}{m^3}$, $\zeta = \frac{24}{n^5} - \frac{24}{m^5}$, $\theta = \frac{720}{n^7} - \frac{720}{m^7}$, &c. Therefore (by art. 832), $\text{S} = \frac{\text{A}}{2e} - \frac{e\beta}{12} + \frac{7e^3\delta}{720} - \frac{31\zeta}{30240} + \&c. = \frac{n \times \log. n - m \times \log. m}{2e} - \frac{n - m}{2e} - \frac{e}{12}$

×

$\times \overline{\frac{1}{n} - \frac{1}{m}} + \frac{7e^3}{360} \times \overline{\frac{1}{n^3} - \frac{1}{m^3}} - \frac{31e^5}{1260} \times \overline{\frac{1}{n^5} - \frac{1}{m^5}} + \frac{127e^7}{1680} \times$
$\overline{\frac{1}{n^7} - \frac{1}{m^7}}$ — &c. And this is the same solution which Mr. *Stirling* derives from his method, *prop.* 28, *De Interpol. Serierum.*

839. The terms in arithmetical progression being represented by $m, m+e, m+2e, m+3e, m+4e, \ldots n-e$, where m denotes the least term, e the common difference, and $n-e$ the greatest term; the sum of the logarithms computed by the theorem for S, in art. 830, is equal to the excess of the series $\overline{\frac{n}{e} - \frac{1}{2}} \times \log. n - \frac{n}{e} + \frac{e}{12n} - \frac{e^3}{360n^3} + \frac{e^5}{1260n^5} -$ &c. above $\overline{\frac{m}{e} - \frac{1}{2}} \times \log. m - \frac{m}{e} + \frac{e}{12m} - \frac{e^3}{360m^3} + \frac{e^5}{1260m^5} -$ &c. For if we now suppose $EA = m$, and $Ea = n$, AF will be the first ordinate, the area $AFfa = n \times \log. n - m \times \log. m$, $\alpha = af - AF = \log. n - \log. m$, the difference of the fluxions of af and AF, or $\beta = \frac{1}{n} - \frac{1}{m}$, $\delta = \frac{2}{n^3} - \frac{2}{m^3}$, &c.; and the theorem appears by substituting these values for A, β, δ, &c. in the equation for S, in art. 830. This coincides with the value of S derived by Mr. *De Moivre* in a different manner, *Suppl. ad Miscel. Analyt.*

840. The sum of the logarithms of the odd numbers, 3, 5, 7, 9, 11, ... $n-1$ is obtained expeditiously, when n is a large number, by computing $\frac{n}{2} \times \log. n - \frac{n}{2} - \frac{1}{12n} + \frac{7}{360n^3} - \frac{31}{1260n^5} + \frac{127}{1680n^7} -$ &c. and thereafter adding $\frac{\log. 2}{2}$, or the constant logarithm .346573590 &c. Because, if we suppose, in art. 838, $e = 1$, and $m = 2$, then $\frac{m \times \log. m}{2} - \frac{m}{2} - \frac{1}{12m} + \frac{7}{360m^3} - \frac{31}{1260m^5} +$ &c. $= \log. 2 - 1 - \frac{1}{12 \times 2} + \frac{7}{360 \times 8} - \frac{31}{1260 \times 32} +$ &c. $=$ (by art. 837) $\log. 2 - \frac{3}{2} \times \log. 2 =$

$-\frac{1}{2}\times\log.2$; and this quantity is to be substracted from $\frac{n}{2}\times\log.n-\frac{n}{2}-\frac{1}{12n}+$ &c. in order to obtain the sum of the logarithms of 3, 5, 7, . . . $n-1$, by art. 838.

841. The sum of a series, of which the terms are alternately positive and negative, is found by computing separately the sums of such as are affected with the same sign by either of the theorems in art. 830 or 832, and then taking the difference of these sums. But when the terms, which are added and substracted alternately, may be considered as the successive ordinates of the same figure, the computation of the area may be avoided, and the sum of the series more elegantly obtained by the following theorem. Let AF represent the first positive term, af the term which when the progression is continued succeeds after the last negative term, e the common distance of the ordinates, S the sum of the terms that precede af, and let β, δ, ζ, &c. denote the differences by which the alternate fluxions of af exceed the respective fluxions of AF, as formerly. Then $S=\frac{AF-af}{2}+\frac{e\beta}{4}-\frac{e^3\delta}{48}+\frac{e^5\zeta}{480}-$ &c. For the sum of the positive terms (by art. 830, the common distance of the ordinates which represent them being $2e$) is $\frac{A}{2e}-\frac{a}{2}+\frac{2e\beta}{12}-\frac{8e^3\delta}{720}+$ &c. and the sum of the negative terms is (art. 832) $\frac{A}{2e}-\frac{e\beta}{12}+\frac{7e^3\delta}{720}-$ &c. the difference of which (a being equal to $af-AF$) is $\frac{AF-af}{2}+\frac{e\beta}{4}-\frac{e^3\delta}{48}+$ &c. If the first, third, and higher alternate fluxions of af vanish, and β, δ, ζ, &c. represent the first, third, and higher alternate fluxions of AF, without changing their signs, then $S=\frac{AF-af}{2}-\frac{e\beta}{4}+\frac{e^3\delta}{48}-\frac{e^5\zeta}{480}+\frac{17e^7\theta}{80640}-$ &c.

842. Hence if $EA=2$, $AF=\log.2$, $Ea=n$, $af=\log.n$, and β, δ, ζ, &c. denote the several fluxions of AF, the logarithm

rithm of the ultimate value of the product $\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9} \times \frac{10}{11} \times \ldots \times \frac{n-2}{n-1} \times 2\sqrt{n}$ will be equal to $\frac{\log. 2 - \log. n}{2} - \frac{\beta}{4} + \frac{\delta}{48} - \frac{\zeta}{480} + \text{\&c.} + \log. 2 + \frac{\log. n}{2} = \frac{3}{2} \times \log. 2 - \frac{\beta}{4} + \frac{\delta}{48} - \frac{\zeta}{480} + \text{\&c.} =$ (because, by art. 837, $\frac{3}{2} \times \log. 2 = 1 + \frac{\beta}{12} - \frac{7\delta}{720} + \frac{31\zeta}{30240} - \text{\&c.}$) $1 - \frac{2\beta}{12} + \frac{8\delta}{720} - \frac{32\zeta}{30240} + \text{\&c.} =$ (because $\beta = \frac{1}{2}$, $\delta = \frac{2}{8}$, $\zeta = \frac{24}{32}$, &c.) $1 - \frac{1}{12} + \frac{1}{360} - \frac{1}{1260} + \frac{1}{1680} - \text{\&c.}$ But (by what has been shown by Dr. *Wallis*) if c denote the circumference of the radius unit, $c = 8 \times \frac{8}{9} \times \frac{24}{25} \times \frac{48}{49} \times \frac{80}{81} \times$ &c. which product continued till the denominator of the last fraction be $\overline{n-1}^2$, may be expressed by $4 \times \frac{4}{9} \times \frac{16}{25} \times \frac{36}{49} \times \frac{64}{81} \times \ldots \times \frac{\overline{n-2}^2}{\overline{n-1}^2} \times n$; consequently $\sqrt{c}$ is the ultimate value of $\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9} \times \ldots \times \frac{n-2}{n-1} \times 2\sqrt{n}$; and $\log. \sqrt{c} = \frac{\log. c}{2} = 1 - \frac{1}{12} + \frac{1}{360} - \frac{1}{1260} + \frac{1}{1680} - \text{\&c.}$ This (which was first observed by Mr. *Stirling*) serves for abridging the computation in finding the sum of the logarithms of the numbers 1, 2, 3, 4, 5, . . . $n-1$. For suppose, in art. 839, $m = e = 1$, then the latter series in that article $\overline{\frac{m}{e} - \frac{1}{2}} \times \log. m - \frac{m}{e} + \frac{e}{12m} - \frac{e^3}{360m^3} + \text{\&c.} = -1 + \frac{1}{12} - \frac{1}{360} + \frac{1}{1260} - \text{\&c.} = \frac{-\log. c}{2}$; consequently $S = \overline{n-\frac{1}{2}} \times \log. n - n + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \text{\&c.} + \frac{\log. c}{2}$; or if $n-\frac{1}{2}$ denote the greatest number in the progression, then (substituting

stituting $\frac{1}{2}$ for e in art. 838) $S = n \times \log. n - n - \frac{1}{24n} + \frac{7}{2880n^3} - \frac{31}{40320n^5} + \&c. + \frac{\log. c}{2}$; which are the rules given for this case in the treatises above-mentioned. But if it is required to find the value of $8 \times \frac{8}{9} \times \frac{24}{25} \times \frac{48}{49} \times$ &c. by the theorem in art. 841 (that is, to compute c from Dr. *Wallis's* proposition), then, because the series for the logarithm of $\sqrt{c}$ converges at too slow a rate, when EA is supposed equal to 2, let r be any greater even number; find the number whose logarithm is $\frac{1}{4r} - \frac{1}{24r^3} + \frac{1}{20r^5} - \&c.$ call this number N, and $\sqrt{c} = 2 \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9} \ldots \times \frac{r-2}{r-1} \times \frac{\sqrt{r}}{N}$. If $r = 10$, then let N be the number whose logarithm is equal to $\frac{1}{40} - \frac{1}{24000} + \frac{1}{2000000} - \&c.$ and $\sqrt{c} = \frac{256}{315} \times \frac{\sqrt{10}}{N}$.

843. In like manner the logarithm of the ultimate value of other products of this kind may be found; as of $3 \times \frac{21}{25} \times \frac{77}{81} \times \frac{165}{169} \times \frac{285}{289} \times$ &c. where the denominators are the squares of the odd numbers 5, 9, 13, 17, 21, &c. whose common difference is 4, and each numerator is less than its denominator by 4. Let the ultimate value of this product be called p, and $\sqrt{p}$ will be the ultimate value of $\frac{3}{5} \times \frac{7}{9} \times \frac{11}{13} \times \frac{15}{17} \times \ldots \frac{n-4}{n-2} \times \sqrt{n}$. Let r be any number in the progression, 3, 7, 11, 15, 21, 25, &c, and N the number whose logarithm is equal to $\frac{1}{2r} - \frac{1}{3r^3} + \frac{8}{5r^5} - \&c.$ then $\sqrt{p} = \frac{\sqrt{r}}{N} \times \frac{3}{5} \times \frac{7}{9} \times \frac{11}{13} \times \frac{15}{17} \ldots \times \frac{r-4}{r-2}$.

844. The problem, concerning the ratio of the sum of all the *unciæ* of the power of a binomial to the *uncia* of the middle term, may be resolved by article 838 or 839, with article 842, or rather by the following theorem. Let r be the exponent of

of the power to which the binomial is to be raised when the exponent is an even number, or equal to this exponent diminished by unit when it is an odd number; and c denote the circumference of the circle when the radius is unit; let N denote the number whose logarithm is equal to $\frac{1}{4\times\overline{r+1}} - \frac{1}{24\times\overline{r+1}^2} + \frac{1}{20\times\overline{r+1}^5} -$ &c. and the ratio required will be that of $\frac{\sqrt{c\times\overline{r+1}}}{2N}$ to unit: for this ratio is equal to $1\times\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\times\frac{8}{9}\times\ldots\times\frac{r-2}{r-1}\times r$, which (by art. 842) is equal to the ultimate value of $\frac{\sqrt{c}}{2}\times\frac{r+1}{r+2}\times\frac{r+3}{r+4}\times\frac{r+5}{r+6}\times\ldots\times\sqrt{\overline{r+s}}$, where s is supposed to represent a number that continually increases by the increment 2; and the logarithm of this ultimate value is (by art. 841, supposing AF = log. $\overline{r+1}$, and af = log. $\overline{r+s}$) $\frac{\log.\sqrt{c}}{2} - \frac{\log.\overline{r+s}}{2} + \frac{\log. r+1}{2} - \frac{1}{4\times\overline{r+1}} + \frac{1}{24\times\overline{r+1}^3} - \frac{1}{20\times\overline{r+1}^5} +$ &c. $+ \frac{\log.\overline{r+s}}{2} =$ log. $\frac{\sqrt{c\times\overline{r+1}}}{2} - \frac{1}{4\times\overline{r+1}} + \frac{1}{24\times\overline{r+1}^3} - \frac{1}{20\times\overline{r+1}^5} +$ &c. $=$ log. $\frac{\sqrt{c\times\overline{r+1}}}{2N}$. These are always supposed to be hyperbolic logarithms, but are converted into tabular logarithms by multiplying by 0.4342944819 &c. The resolution of this problem derived from other principles may be found in Mr. *De Moivre's Suppl. Miscel. Analyt. p.* 17, and Mr. *Stirling's Tract. de Summat. Serier. p.* 119. Because the other coefficients of the terms of a binomial (when the exponent r is an even number) are found by multiplying the coefficient of the middle term by $\frac{r}{r+2}\times\frac{r-2}{r+4}\times\frac{r-6}{r+6}\times$ &c. these may be likewise found by art. 838, &c. For the use of the properties of the terms of a binomial, when raised to a high power, see Mr.

James Bernouilli's Ars. Conject. part 4, chap. 4, and Mr. *De Moivre's Doctrine of Chances.*

845. The sum of the series $\frac{1}{m^r} - \frac{1}{\overline{m+e}|^r} + \frac{1}{\overline{m+2e}|^r} - \frac{1}{\overline{m+3e}|^r} + \frac{1}{\overline{m+5e}|^r}$ — &c. is (by art. 841, because *af* with all its fluxions ultimately vanish in this case) $\frac{1}{2m^r} + \frac{reA}{2m} - \frac{\overline{r+1} \times \overline{r+2}}{12mm} \times eeB + \frac{\overline{r+3} \times \overline{r+4}}{10mm} \times eeC - \frac{17\,\overline{r+5} \times \overline{r+6}}{168mm} \times eeD + \&c.$ where A in the usual manner denotes the first term, $\frac{1}{2m^r}$, B the second, C the third, not including its sign, &c. If $r = 1$, then $S = \frac{1}{2m} + \frac{1}{4mm} - \frac{1}{8m^4} + \frac{1}{4m^6} - \frac{17}{8m^8} + \&c.$ And hence the sum of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$ &c. (which is equal to the hyperbolic logarithm of 2) may be easily computed to a great number of decimal places, by first collecting the sum of the terms at the beginning of the series (by common arithmetic) that precede $\frac{1}{m}$, so as that m may be a pretty large number (equal to 25 or 27, for example), and then computing the sum of the other terms by this series. If $r = 2$, then $S = \frac{1}{2mm} + \frac{1}{2m^3} - \frac{1}{2m^5} + \frac{3}{2m^7} -$ &c. whence the sum of the series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{49} +$ &c. may be computed in like manner. By supposing $r = 1$ and $e = 2$, the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} +$ &c. may be computed by first collecting the sum of the terms at the beginning of the series that precede $\frac{1}{m}$, viz. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots - \frac{1}{m-2}$, by common arithmetic; and then adding $\frac{1}{2m} + \frac{1}{2mm} - \frac{1}{m^4} + \frac{8}{m^6} - \frac{34}{m^8} +$ &c. This series is equal to $\frac{1}{8}$ of the circumference of the circle, the radius being equal to 1, by art. 746; and hence the ratio of the circumfe-

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rence to the diameter may be computed to many decimal places with little labour.

846. The theorems in art. 830 and 832 may be applied for approximating to the sum of the series that is formed by substituting successively any numbers in arithmetical progression in the place of x in the fraction $\frac{1}{\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \&c.}$ (where a, b, c, &c. represent any given numbers), by what was shown in the last chapter concerning the area, when the ordinate is equal to such a fraction, &c. And in some cases this sum may be assigned accurately by art. 361.

847. If N represent the number whose hyperbolic logarithm is e, the sum of the series $1 + \frac{e}{2} + \frac{e^2}{12} - \frac{e^4}{720} + \frac{e^6}{30240} - \frac{e^8}{1209600} + \&c. = \frac{eN}{N-1}$: and the sum of the series $\frac{1}{2} - \frac{e^2}{12} + \frac{7e^4}{720} - \frac{31e^6}{30240} + \frac{127e^8}{1209600} - \&c. = \frac{eN}{NN-1}$. These appear by supposing, in art. 830 and 832, the curve FMf to be the logarithmic, Aa its asymptote, AF or RV equal to the *modulus*, and finding the sum of the ordinates by the common rule for a geometrical progression, and putting this sum equal to the value of S in those articles.

848. The base Aa being divided into any number of equal parts represented by n, let the area AFfa = Q, the sum of the extreme ordinates AF + af = A, the sum of all the intermediate ordinates BE + CK + &c. = B, the base Aa = R, and the same quantities be represented by β, δ, ζ, &c. as formerly; then the area AFfa = Q = $\overline{\frac{A}{2n+2} + \frac{nB}{nn-1}} \times R - \frac{R^4\delta}{720nn} + \frac{R^6\zeta}{30240} \times \frac{nn+1}{n^4} - \&c.$ For supposing, in art. 830, $e = \frac{R}{n}$, S + $\frac{af - AF}{2} = B + \frac{A}{2} = \frac{nQ}{R} + \frac{R\beta}{12n} - \frac{R^3\delta}{720n^3} + \frac{R^5\zeta}{30240n^5} - \&c.$ and supposing $e = R$ in the same theorem, $\frac{AF + af}{2} = \frac{A}{2} = \frac{Q}{R} + \frac{R\beta}{12} - \frac{R^3\delta}{720} + \frac{R^5\zeta}{30240} - \&c.$ then, if we exterminate β

by

by these two equations, the proposition will appear. If we neglect δ, ζ, θ, &c. AF*fa* $= \frac{AR}{2n+2} + \frac{nBR}{nn-1}$. Suppose $n = 2$, or that there are three ordinates only (in which case B denotes the middle ordinate), then the area AF*fa* $= \frac{A+4B}{6} \times R - \frac{R^4\delta}{4\times720} + \frac{5R^6\zeta}{16\times30240}$ —&c. If we suppose $n=3$, or that there are four ordinates only, B will represent the sum of the second and third, and the area AF*fa* $= \frac{A+3B}{8} \times R - \frac{R^4\delta}{9\times720} + \frac{R^6\zeta}{81\times3024}$ — &c. By neglecting δ, ζ, θ, &c. we shall have two of the theorems given by Sir *Isaac Newton* and others for computing the area from equidistant ordinates, the latter of which (viz. AF*fa* $= \frac{A+3B}{8} \times R$) is much recommended by Mr. *Cotes*.

849. By exterminating δ, ζ, θ, &c. successively, other theorems will be found by which the area will be more and more accurately determined from the ordinates. Let there be five ordinates, A the sum of the first and last, B the sum of the second and fourth, and C the middle ordinate; then the area AF*fa* $= \frac{7A+32B+12C}{90} \times R - \frac{31R^6\zeta}{6\times16\times16\times30240} +$ &c.; for by the rule for three ordinates $\frac{Q}{R} = \frac{A+4C}{6} \times R - \frac{R^3\delta}{4\times720} + \frac{5R^5\zeta}{16\times30240}$. By dividing the base into two equal parts, and computing from the same rule the area that stands upon each part, and adding these areas together, $\frac{2Q}{R} = \frac{A+4B+2C}{6} - \frac{R^3\delta}{32\times720} + \frac{5R^5\zeta}{2\times16\times16\times30240}$ — &c. then, by exterminating δ by these two equations, the proposition appears. These theorems may be continued in like manner, and some judgment formed of the accuracy of the several rules, by comparing the quantities that are neglected in them.

850. The theorems in art. 830 and 832, from which we have drawn so many conclusions, may be of use for interpolating the terms of a series likewise, or for finding the intermediate ordinates of a figure when the equidistant primary ordinates are given. When the equation of the figure FM*f* is known, the intermediate ordinates are found without any difficulty, by substituting the intermediate values of the base in the equation; but it is not so obvious how we are to interpolate the values of S, or the sums of those ordinates. Suppose FN*z* (*fig.* 317) to be the figure whose successive ordinates at the points A,B,C,D,&c. are always equal to the successive sums of the ordinates of the figure FM*f* at the same points beginning with AF; that is, let AF = AF, $B\epsilon$ = AF + BE, $C\varkappa = B\epsilon$ + CK, $D\lambda = C\varkappa$ + DL, &c. and let it be required to determine any intermediate ordinate PN of the figure FN*z*. Let this ordinate PN meet the curve FM*f* in M, AF $= a$, PM $= y$, the common distance of the ordinates AB $= e$, the area AFMP = Q, $\dot{y} - \dot{a} = \beta$, $\dddot{y} - \dddot{a} = \delta$, &c. then PN $= \frac{Q}{e} + \frac{a+y}{2} + \frac{e\beta}{12} - \frac{e^3\delta}{720} + \frac{e^5\zeta}{30240} - \frac{e^7\theta}{1209600}$ + &c. because, if we suppose PN to move successively into the places of the ordinates of the figure FN*z* at A, B, C, D, &c. its successive values will be rightly determined by this theorem, by art. 830. Or if we would avoid the area Q in the theorem for PN, let AP $= m$, and, since $\frac{a+y}{2} = \frac{Q}{m} + \frac{m\beta}{12} - \frac{m^3\delta}{720} + \frac{m^5\zeta}{30240}$ − &c. it follows, that PN $= \frac{a+y}{2} \times \frac{e+m}{e} + \frac{e^2-m^2}{12e} \times \beta - \frac{e^4-m^4}{720e} \times \delta + \frac{e^6-m^6}{30240e} \times \zeta$ − &c. A similar theorem follows from art. 832: let AR be taken backwards from A, and P*r* forwards from P, each equal to ½AB, RV and *rv* meet FM*f* in V and *v*; let Q now denote the area RV*vr*, and β, δ, ζ, &c. denote the differences by which the first, third, fifth, and higher alternate fluxions of the ordinate *rv*, exceed the respective fluxions of RV, and AB $= e$, as

formerly; then any ordinate $PN = \frac{Q}{e} - \frac{e\beta}{2\times 12} + \frac{7e^3\delta}{8\times 720} - \frac{31e^5\zeta}{32\times 30240} +$ &c. The ordinates at the points A, B, C, D, &c. are called the primary ordinates of the figures FNz or FMf. If $Pp = AB$, pn meet FNz in n and FMf in m, then $pn = PN + pm$ or $PN - pm$, according as Pp is taken forwards or backwards from P: and hence any intermediate ordinate PN being known, all other ordinates of the figure FNz that are at a distance from it equal to AB, or any multiple of AB, are easily found by adding or substracting the intermediate ordinates of the figure FMf.

851. Let TX and T′ X′ be the primary ordinates of the figure FMf adjoining to the intermediate ordinate PN; bisect TT′ in x, let the ordinate xy meet FMf in y, the area $xyvr = q$, the ordinate at T of the figure FNz, viz. $T\tau = s$, $xy = y$, $rv = u$, then $PN = s + \frac{q}{e} - \frac{e}{2\times 12} \times \overline{\dot u - \dot y} + \frac{7e^3}{8\times 720} \times \overline{\dddot u - \dddot y} -$ &c. For if $RV = a$, the area $RVvr = Q$, then $RVyx = Q - q$, and $PN = \frac{Q}{e} - \frac{e}{2\times 12} \times \overline{\dot u - \dot a} + \frac{7e^3}{8\times 720} \times \overline{\dddot u - \dddot a} -$ &c. by the last article; and $T\tau = s = \frac{Q-q}{e} - \frac{e}{2\times 12} \times \overline{\dot y - \dot a} + \frac{7e^3}{8\times 720} \times \overline{\dddot y - \dddot a} -$ &c. by art. 830; consequently $PN - s = \frac{q}{e} - \frac{e}{24} \times \overline{\dot u - \dot y} + \frac{7e^3}{8\times 720} \times \overline{\dddot u - \dddot y} -$ &c. and $PN = s + \frac{q}{e} - \frac{e}{24} \times \overline{\dot u - \dot y} +$ &c. This series will converge very fast in many cases, when PN is at a great distance from AF.

852. This leads us to some easy and simple theorems for finding the intermediate terms of a series by interpolating the differences of the terms. First, suppose the differences of the terms to decrease continually, so that by continuing the series these differences may become less than any given quantity, but never vanish; or that the terms of the series being represented by the ordinates of the figure FNz, and their differences by the ordinates of FMf, this latter figure has the base Ff for its asymptote.

tote. In this case $\pi\nu$ the term of the series that precedes the first primary term AF, at any distance $A\pi$ less than AB, is equal to the excess of the sum of the primary differences AF + BE + CK + &c. above the sum of the interpolated differences $be + ck + dl$ + &c. the distances B*b*, C*c*, D*d*, &c. being each equal to $A\pi$, and taken the same way from B, C, D, &c. For in this case PN is ultimately equal to $T\tau$; that is (supposing PT $= \pi$A), $\pi\nu + be + ck + dl$ + &c. is ultimately equal to AF + BE + CK + &c.; consequently $\pi\nu$ = AF — be + BE — ck + CK — dl + &c.

853. For example, suppose AF = 1, BE $= \frac{1}{2}$, CK $= \frac{1}{3}$, DL $= \frac{1}{4}$, &c. then the successive primary ordinates of the figure FN*z* will be $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{147}{60}$, &c. Let $A\pi = \frac{1}{2}$ AB, and because the intermediate differences $be = \frac{2}{3}$, $ck = \frac{2}{5}$, $dl = \frac{2}{7}$, &c. it follows, that $\pi\nu = 1 - \frac{2}{3} + \frac{1}{2} - \frac{2}{5} + \frac{1}{3} - \frac{2}{7} + \frac{1}{5} - \frac{2}{9}$ + &c. $= 2 \times \overline{\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7}}$ + &c. = (because log. $2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$ + &c.) $2 \times \overline{1 - \log. 2}$. And the other intermediate terms are found by adding successively the intermediate differences $\frac{2}{3}, \frac{2}{5}, \frac{2}{7}$, &c.*

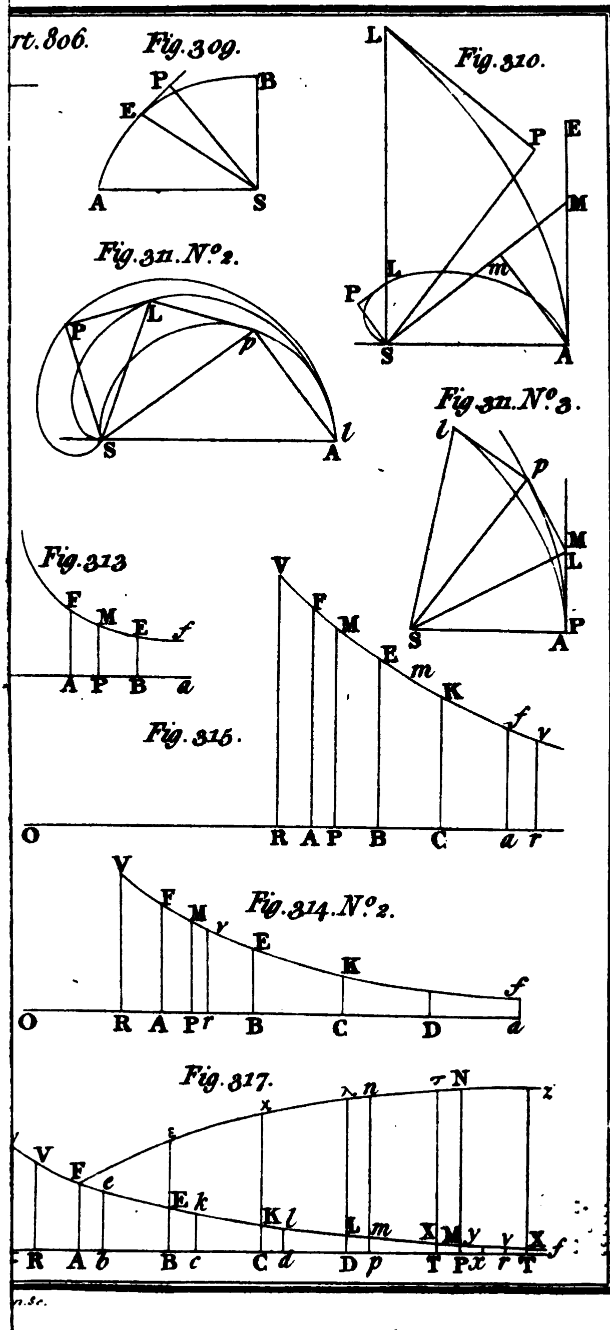
854. In like manner, if we suppose AF = 1, BE $= \frac{1}{4}$, CK $= \frac{1}{9}$, DL $= \frac{1}{16}$, &c. or the successive primary ordinates of FM*f* to be the squares of $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, &c. and we suppose $A\pi = \frac{1}{2}$ AB, then the intermediate differences be, ck, dl, &c. will be $\frac{4}{9}, \frac{4}{25}, \frac{4}{49}$, &c. the ordinates AF, Bς, C$\varkappa$, Dλ, &c. will be

* The intermediate terms of this series are determined by the learned Mr. *Euler*, *Comment. Petropol. tom.* 5, *p.* 93, by finding a fluent that expresses the terms of the series in a general manner, which in this case is F, $\frac{1 - x^n}{1 - x} \times \dot{x}$, supposing n to denote the place of the term in the series (that is, 1 for the first term, 2 for the second, &c. and $\frac{1}{2}$ for the term $\pi\nu$), and 1 to be substituted for x after the fluent is determined; whence $\pi\nu$ = F, $\frac{1 - \sqrt{x}}{1 - x} \times \dot{x} = \overline{2 - 2} \times \log. 2$. I take this opportunity to mention, that, having occasionally shown, in 1737, the 292 and 293d pages of this Treatise (after they were printed) to Mr. *Stirling*, he took notice that a theorem similar to the first of these described in art. 352 had been communicated to him by Mr. *Euler*.

$1, 1 + \frac{1}{4}, 1 + \frac{1}{4} + \frac{1}{9}, 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$, &c. and the ordinate πv that stands before the first primary ordinate AF, at half the common distance of the ordinates, will be equal to $1 - \frac{4}{9} + \frac{1}{4} - \frac{4}{25} + \frac{1}{9} - \frac{4}{49} + \&c. = 4 \times \overline{\frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25}} + \&c.$ Therefore, if the sum of the series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{49} + \&c.$ (which may be computed easily from art. 845) be denoted by N, then $\pi v = 4 - 4N$. If $AF = 1$, $BE = \frac{1}{5}$, $CK = \frac{1}{9}$, $DL = \frac{1}{13}$, &c. and $A\pi = \frac{1}{2}AB$, then $\pi v = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.$ which is equal to the eighth part of the circumference of the radius unit.

855 (*Fig.* 318). When the terms may be continued without end, and their second differences decrease so as ultimately to vanish, let K denote the ultimate value of the first differences of the terms; and πv will be equal to $\frac{AB - A\pi}{AB} \times K$ added to the excess of the sum of the primary differences $AF + BE + CK$, &c. above the sum of the intermediate differences $be + ck + dl + \&c.$ because in this case the fluxions of rv and xy ultimately vanish, PN is ultimately equal to $f + \frac{q}{e}$, q to $K \times xr = K \times \overline{AB - A\pi}$, and consequently $\pi v = \frac{AB - A\pi}{AB} \times K + AF - be + BE - ck + \&c.$ A like theorem may be applied, when the second differences of the terms continually approach to a certain limit.

856. The series $1, 1 \times 1, 1 \times 2, 1 \times 2 \times 3, 1 \times 2 \times 3 \times 4, 1 \times 2 \times 3 \times 4 \times 5$, &c. being proposed, let it be required to find the term that is betwixt the two first primary terms at equal distances from each. The differences of the logarithms of the terms are log. 1, log. 2, log. 3, log. 4, log. 5, &c. and the ordinates of the figure FMf being supposed to represent these logarithms, the intermediate ordinates will be log. $\frac{3}{2}$, log. $\frac{5}{2}$, log. $\frac{7}{2}$, log. $\frac{9}{2}$, &c. Therefore the logarithm of the term required is $\frac{K}{2}$ + log. 1 — log. $\frac{3}{2}$ + log. 2 — log. $\frac{5}{2}$ + log.


rt. 806.
Fig. 309.
Fig. 310.
Fig. 311. N°. 2.
Fig. 311. N°. 3.
Fig. 313
Fig. 315.
Fig. 314. N°. 2.
Fig. 317.

log. 3 — log. $\frac{7}{2}$ + &c. which is equal to the logarithm of the ultimate value of $\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9} \ldots \times \frac{n}{n-1} \times \sqrt{n+1} =$ (by what was shown in art. 842, from Dr. *Wallis*) log. $\frac{\sqrt{c}}{2}$; consequently the term required is equal to half the square root of the circumference of the radius 1; which is agreeable to what has been discovered by other methods. This subject might be prosecuted further, and other instances given of the use of the method of fluxions in finding the sum of a series, or interpolating its terms; but we proceed to what is more necessary for bringing this Treatise to a proper conclusion.

CHAP. V.

Of the general Rules for the Resolution of Problems.

857 (*Fig.* 319). IT remains that we describe briefly the general rules that are derived from this method for the resolution of problems, and illustrate them by examples. The base AP being represented by x, and the ordinate PM by y, the *subtangent* PT (which is the right line intercepted upon the base betwixt the ordinate and tangent) is found by computing $\frac{y\dot{x}}{\dot{y}}$. When y increases while x increases, this value of PT is positive, and PT is on the same side of P with PA; but when y decreases while x increases, this value of PT is negative, and PT is on the other side of P. If $\dot{x}$ vanish in respect of $\dot{y}$, PT vanishes, and the ordinate is the tangent; but if $\dot{y}$ vanish in respect of $\dot{x}$, the tangent is parallel to the base. If the curve FM be represented by z, then the tangent MT $= \frac{y\dot{z}}{\dot{y}} = \frac{y\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{y}}$. If MN perpendicular to the tangent MT meet the base in N, PN (which is sometimes called the *subnormal*) $= \frac{y\dot{y}}{\dot{x}}$. These fol-

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low from art. 188, &c. by which $\dot{x}$, $\dot{y}$, and $\dot{z}$, are in the same proportion as the right lines PT, PM, and MT; or as PM, PN, and MN. For example, if $y^m = a^{m-1}x$, then (art. 728, *fig.* 320) $\frac{m\dot{y}}{y} = \frac{\dot{x}}{x}$, and $PT = \frac{y\dot{x}}{\dot{y}} = mx = m \times AP$. Let the *ray* SE revolve about a given centre S, and meet the curve AEB in E, the ark fE be described from the centre S, SE $= r$, the ark of the curve AE $= s$, the fluxion of the circular ark fE be represented by $\dot{z}$, and SP be perpendicular from S on the tangent EP in P; then $SP = \frac{r\dot{z}}{\dot{s}}$, and $EP = \frac{r\dot{r}}{\dot{s}}$, by art. 202. There are other theorems relating to the tangents which are of use in particular enquiries, of which some were given in *book* I. *chap.* 8.

858. When the first fluxion of the ordinate vanishes, if at the same time its second fluxion is positive, the ordinate is then a *minimum*, but is a *maximum* if its second fluxion is then negative; that is, it is less in the former, and greater in the latter case than the ordinates from the adjoining parts of that branch of the curve on either side. This follows from what was shown at great length in *chap.* 9, *b.* I., or may appear thus. Let the ordinate AF $=$ E, AP $= x$ (*fig.* 319), and, the base being supposed to flow uniformly, the ordinate PM $=$ (art. 751) $E + \frac{\dot{E}x}{\dot{x}} + \frac{\ddot{E}x^2}{2\dot{x}^2} + \frac{\dddot{E}x^3}{6\dot{x}^3} +$ &c.; let Ap be taken on the other side of A equal to AP, then the ordinate $pm = E - \frac{\dot{E}x}{\dot{x}} + \frac{\ddot{E}x^2}{2\dot{x}^2} - \frac{\dddot{E}x^3}{6\dot{x}^3} +$ &c. Suppose now $\dot{E} = o$, then $PM = E_{*} + \frac{\ddot{E}x^2}{2\dot{x}^2}$ &c. and $pm = E_{*} + \frac{\ddot{E}x^2}{2\dot{x}^2}$ —&c. Therefore if the distances AP and Ap be small enough, PM and pm will both exceed the ordinate AF when $\ddot{E}$ is positive; but will be both less than AF

AF if $\ddot{E}$ be negative. But if $\ddot{E}$ vanish as well as $\dot{E}$, and $\dddot{E}$ does not vanish, one of the adjoining ordinates PM or *pm* shall be greater than AF, and the other less than it; so that in this case the ordinate is neither a *maximum* nor *minimum*. We always suppose the expression of the ordinate to be positive.

859. In general, if the first fluxion of the ordinate, with its fluxions of several subsequent orders, vanish, the ordinate is a *minimum* or *maximum*, when the number of all those fluxions that vanish is 1, 3, 5, or any odd number. The ordinate is a *minimum* when the fluxion next to those that vanish is positive; but a *maximum* when this fluxion is negative. This appears from art. 261, or by comparing the values of PM and *pm* in the last article. But if the number of all the fluxions of the ordinate of the first and subsequent successive orders that vanish be an even number, the ordinate is then neither a *maximum* nor *minimum*.

860. When the fluxion of the ordinate y is supposed equal to nothing, and an equation is thence derived for determining x, if the roots of this equation are all unequal, each gives a value of x that may correspond to a greatest or least ordinate. But if two, or any even number of these roots be equal, the ordinate that corresponds to them is neither a *maximum* nor *minimum*. If an odd number of these roots be equal, there is one *maximum* or *minimum* that corresponds to these roots, and one only. Thus if $\frac{\dot{y}}{\dot{x}} = x^4 + ax^3 + bx^2 + cx + d$, then, supposing all the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = o$ to be real, if the four roots are equal, there is no ordinate that is a *maximum* or *minimum*; if two or three of the roots only are equal, there are two ordinates that are *maxima* or *minima*; and if all the roots are unequal, there are four such ordinates.

861. To give a few examples of the most simple cases. Let $y = a^2x - x^3$, then $\dot{y} = a^2\dot{x} - 3x^2\dot{x}$ and $\ddot{y} = -6x\dot{x}^2$. Suppose $\dot{y} = o$, and $3x^2 = a^2$ or $x = \frac{a}{\sqrt{3}}$, in which case $\ddot{y} =$ —

$-\frac{6a\dot{x}^2}{\sqrt{3}}$. Therefore, $\ddot{y}$ being negative, y is a *maximum* when $x = \frac{a}{\sqrt{3}}$, and its greatest value is $\frac{2a^3}{3\sqrt{3}}$. If $y = aa + 2bx - xx$, then $\dot{y} = 2b\dot{x} - 2x\dot{x}$, and $\ddot{y} = -2\dot{x}^2$; consequently y is a *maximum* when $2b - 2x = o$, or $x = b$. If $y = aa - 2bx + xx$, then $\dot{y} = -2b\dot{x} + 2x\dot{x}$, and $\ddot{y} = 2\dot{x}^2$; consequently y is now a *minimum* when $x = b$, if a be greater than b.

862. The right lines BF and GH (*fig.* 321) being perpendicular to the given right line BG in the same plane, H a given point, C any point upon BF, and the figure being supposed to revolve about the axis BG, let it be required to determine the position of the right line HC when the conical surface described by it is a *minimum*. Let DE bisect BG perpendicularly in D, and meet HC in E, and (by art. 216) the surface described by HC about the axis BG will be as DE $\times$ EH $=$ (supposing GH $= a$, DG $= b$, DE $= x$, and consequently EH^2 $= bb + \overline{a-x}^2$) $x\sqrt{bb+aa-2ax+xx}$. Suppose, therefore, $y = b^2x^2 + a^2x^2 - 2ax^3 + x^4$, then $\dot{y} = 4x^3\dot{x} - 6ax^2\dot{x} + 2a^2x\dot{x} + 2b^2x\dot{x}$, and $\ddot{y} = 12x^2\dot{x}^2 - 12ax\dot{x}^2 + 2a^2\dot{x}^2 + 2b^2\dot{x}^2$. By supposing $\dot{y} = o$, we have $\overline{4xx - 6ax + 2aa + 2bb} \times x = o$; the resolution of which equation gives (besides $x = o$) $x = \frac{3a + \sqrt{aa-8bb}}{4}$ or $x = \frac{3a - \sqrt{aa-8bb}}{4}$. If GH $= \sqrt{2} \times$ BG, then $aa = 8bb$, and these two values of x become equal to each other and to $\frac{3a}{4}$. In this case $\frac{\ddot{y}}{\dot{x}^2} = 12 \times \frac{3a}{4} \times \frac{-a}{4} + 2aa + \frac{aa}{4} = o$, and y is neither a *maximum* nor *minimum*; but while we suppose the point C to move from B along the right line BF, the conical surface described by the right line HC about the axis BG continually increases, though its fluxion vanishes when DE $= \frac{3\text{GH}}{4}$. If GH be greater than $\sqrt{2} \times$ BG, then $aa > 8bb$, the former value of x gives $\ddot{y}$ positive, and y a *minimum*;

mum; but the latter value of x gives $\ddot{y}$ negative, and y a *maximum;* that is, the value of y is greater when $x = \frac{3a - \sqrt{aa-8bb}}{4}$ than its adjoining values on either side. But this is not to be understood as if the value of y was then the greatest possible; for it is obvious that, by supposing the point C to proceed in the right line BF, DE × EH may exceed any given rectangle. See art. 239. When GH is less than $\sqrt{2}$ × BG, aa is less than $8bb$, and the values of x are imaginary. Examples of this kind may frequently occur; and what has been shown of ordinates is transferred to the rays that are drawn from a given point to a curve, by art. 277.

863. When $\dot{y} = o$, if $\ddot{y}$ be at the same time infinite in respect of $\dot{x}$ (which is supposed constant), we cannot conclude that y is then a *maximum* or *minimum* without some further enquiry; for the ordinate may then pass through a point of contrary flexure or a cuspid. Let $\frac{\dot{y}}{\dot{x}} = \frac{\sqrt{ax-xx}}{a}$, then $\frac{\ddot{y}}{\dot{x}^2} = \frac{a-2x}{2a\sqrt{ax-xx}}$. The supposition of $\dot{y} = o$ gives $ax - xx = o$, and $x = a$, or $x = o$; in both cases $\ddot{y}$ is infinite; and it is obvious that the curve is reflected from the ordinate, because when x is supposed greater than a, or negative, the values of $\dot{y}$ are imaginary. In like manner, if $\dot{y}$, $\ddot{y}$, and $\dddot{y}$, vanish, and $\ddddot{y}$ be infinite in respect of $\dot{x}$, we cannot thence conclude y to be a *maximum* or *minimum.* But it may be admitted as a rule, that when $\dot{y}=o$, and, $\dot{x}$ being constant, $\ddot{y}$ is real and finite; or when any odd number of fluxions of y of the successive orders $\dot{y}$, $\ddot{y}$, $\dddot{y}$, &c. vanish together, and the fluxion of the next order to these is real and finite in respect of $\dot{x}$, we may safely conclude (without any further enquiry) that y is then a *maximum* or *minimum,* according as this last fluxion is negative or positive. However, when, after supposing $\dot{y} = o$, x is determined by a simple

simple equation, we may conclude y to be a *maximum* or *minimum* without farther trouble.

864. It was observed in art. 244, that when any quantity N is expressed by a fraction $\frac{P}{Q}$, if P and Q vanish at the same time, we are not thence to conclude that $N = o$. Thus, suppose $N = \frac{aa - ax}{a - \sqrt{ax}}$; and when $x = a$, the numerator and denominator of N vanish together; but if we reduce the value of N to a more simple form, by dividing the numerator and denominator by their common divisor $\sqrt{a} - \sqrt{x}$, we shall find $N = a \times \frac{\sqrt{a} + \sqrt{x}}{a} =$ (when $x = a$) $a \times \frac{2\sqrt{a}}{\sqrt{a}} = 2a$. In such cases the value of N is found by computing $\frac{\dot{P}}{\dot{Q}}$; because when P and Q decrease till they vanish, the ultimate ratio of P to Q is that of $\dot{P}$ to $\dot{Q}$. If $\dot{P}$ and $\dot{Q}$ vanish at the same time, then $N = \frac{\ddot{P}}{\ddot{Q}}$. This rule was given in the *Anal. des Infiniment Petits*, p. 145, and is sometimes of use in preventing mistakes concerning the greatest and least ordinates (as are described *Mem. de l'Acad. des Sciences*, 1706), as well as on other occasions. The computations in enquiries of this kind are sometimes abridged by art. 730. Thus if $m\dot{x}x = ny\dot{y} + \overline{mn - 1} \times \dot{y}x$, then $\overline{y + mx} \times \overline{\dot{x} - n\dot{y}} = o$, and $\overline{\dot{y} + m\dot{x}} \times \overline{\ddot{x} - n\ddot{y}} = o$.

865. The greatest and least ordinates are likewise discovered, in some cases, by supposing $\dot{y}$ to be infinite in respect of $\dot{x}$; but it is obvious that there are several exceptions to this rule, since the curve may then form a continued arch that is reflected from the ordinate after touching it, or may be continued on the other side with a contrary flexure. See art. 262. By comparing the signs of $\dot{y}$ on the different sides of the ordinate (which in this case is a tangent to the curve), the latter of these cases may be distinguished from that wherein the ordinate is a *maximum* or *minimum*; and when the curve is reflected from the

the ordinate, some of the values of $\dot{y}$ become imaginary on one side of that ordinate. As for the *maxima* and *minima*, which were said to be of the second kind in art 240, see art. 276.

866. The points of contrary flexure and reflexion are usually determined by supposing $\ddot{y} = o$ or infinite. But this rule being liable to several exceptions, it was shown, in art. 263, that the ordinate y passes through a point of contrary flexure, when, the curve being continued on both sides of the ordinate, $\dot{y}$ is a *maximum* or *minimum*; which (by what has been shown) does not always happen when $\ddot{y} = o$ or infinite. Hence, if $\ddot{y} = o$, and $\dddot{y}$ be real and finite, then y passes through a point of contrary flexure (*fig.* 319). This appears likewise by comparing the values of PM and pm in art. 859. Let PM meet the tangent at F in V, and pm meet it in v; then $\mathrm{PV} = \mathrm{E} + \frac{\dot{\mathrm{E}}x}{\dot{x}}$, and $pv = \mathrm{E} - \frac{\dot{\mathrm{E}}x}{\dot{x}}$; but when $\ddot{\mathrm{E}} = o$, $\mathrm{PM} = \mathrm{E} + \frac{\dot{\mathrm{E}}x}{\dot{x}} * + \frac{\dddot{\mathrm{E}}x^3}{\dot{x}^3} + \&c.$ and $pm = \mathrm{E} - \frac{\dot{\mathrm{E}}x}{\dot{x}} * - \frac{\dddot{\mathrm{E}}x^3}{\dot{x}^3} + \&c.$; consequently, if $\dddot{\mathrm{E}}$ be positive, and the distances AP and Ap small enough, PM will be greater than PV, and pm less than pv; and whether $\dddot{\mathrm{E}}$ be positive or negative, the arks FM and Fm shall be on different sides of the tangent TFt; consequently F will be a point of contrary flexure: but if $\dddot{\mathrm{E}}$ likewise vanish, and $\ddddot{\mathrm{E}}$ be of a real value, PM and pm will be both greater or both less than the respective perpendiculars PV and pv intercepted by the tangent, and there will be no point of contrary flexure at F. In general, if $\ddot{y}$, $\dddot{y}$, $\ddddot{y}$, &c. vanish, the number of these fluxions being odd, and the fluxion of the next order to them have a real and finite value, then y passes through a point of contrary flexure; but if the number of these fluxions that vanish be even, it cannot be said to pass through such a point, unless it should be allowed that a double infinitely small flexure can be formed

at

at one point. To give one of the most simple examples, suppose $y = 1 - x^4$, then $\dot{y} = -4x^3\dot{x}$, $\ddot{y} = -12x\dot{x}^2$, $\dddot{y} = -24x\dot{x}^3$, and $\ddddot{y} = -24\dot{x}^4$. If we suppose $\ddot{y} = 0$, then $x = 0$; but, because $\dddot{y}$ is then likewise nothing, and $\ddddot{y}$ real and finite, y does not pass through a point of contrary flexure, but is indeed a *maximum*; the truth of which might easily be shown otherwise.

867. The curve being supposed to be continued from the ordinate PM, or y, on both sides, if $\ddot{y}$ be infinite, M is not therefore always a point of contrary flexure, as $\dot{y}$ is not in this case always a *maximum* or *minimum*, by art. 865, and the curve may have its concavity turned the same way on both sides of M. But these cases may be likewise distinguished by comparing the signs of $\ddot{y}$ on the different sides of PM, for, when these signs are different, M is a point of contrary flexure: for example, let $y = 1 - x^{\frac{5}{3}}$, then $\ddot{y} = -\frac{10\dot{x}^2}{9\sqrt[3]{x}}$, which becomes infinite when $x = 0$ or $y = 1$, and is affected with contrary signs on different sides of y; consequently the ordinate passes through a point of contrary flexure when $x = 0$. The suppositions of $\dot{y} = 0$ or infinity, and of $\ddot{y} = 0$ or infinity, serve to direct us where we are to search for the *maxima* or *mimima*; and for points of contrary flexure, but where we are not always sure to find them; for though an ordinate or a fluxion that is positive never becomes negative at once, but by decreasing or increasing gradually (as was shown in art. 262), yet, after it has decreased till it vanish, it may thereafter increase, continuing still positive; or, after increasing till it becomes infinite, it may thereafter decrease, without changing its sign.

868. The points of reflexion, or cuspids, were distinguished into two kinds in art. 268. When the curve is reflected from the ordinate PM or y, it always forms a cuspid, unless when $\dot{y}$ is infinite in respect of $\dot{x}$, in which case likewise M is sometimes a cuspid of the second kind; and when $\ddot{y}$ or $\dddot{y}$ is real and finite, M is always a cuspid of the second kind. If $\ddot{y} = 0$, the

the cuspid may be of either kind. But the most simple kind of the cuspids of the first sort (such as are in some of the lines of the third order) are formed when $\ddot{y}$ is infinite, as the most simple kind of points of contrary flexure are formed where $\ddot{y} = 0$: see art. 270 and 379. When $\dot{y}$ is such a *maximum* or *minimum* as was described in art. 865, y passes through a cuspid of the first kind. Other observations may be derived from art. 269.

869. Suppose (as in art. 857) ST (*fig.* 322) perpendicular from the given point S on the tangent PT in T, SP $= r$, the fluxion of the curve equal to $\dot{s}$, the fluxion of the ark fP described from the centre S equal to $\dot{z}$; consequently ST $= \frac{r\dot{z}}{\dot{s}}$; and (by art. 281) P is a point of contrary flexure, when the angle SPT is oblique, and ST is a *maximum* or *minimum*; whence rules may be deduced analogous to the former for determining those points. Suppose $\dot{z}$ constant, and, the fluxion of ST being equal to $\frac{\dot{z}\dot{r}\dot{s} - r\ddot{s}\dot{z}}{\dot{s}^2}$, the points of contrary flexure are found by supposing $\dot{r}\dot{s} - r\ddot{s}$, or (because $\dot{s}\dot{s} = \dot{r}\dot{r} + \dot{z}\dot{z}$ and $\dot{s}\ddot{s} = \dot{r}\ddot{r}$) $\dot{r}^2 - r\ddot{r}$, equal to nothing or infinity; but with exceptions similar to those described in art. 866 and 867.

870. Let C (*fig.* 319) be the centre of the curvature at M, Cb perpendicular to PM in b, AP$=x$, PM$=y$, the ark FM$=s$, and (by art. 382), supposing $\dot{x}$ constant, M$b = \frac{\dot{s}^2}{\ddot{y}} = \frac{\dot{x}^2+\dot{y}^2}{\ddot{y}}$, or (because $\dot{s}\ddot{s} = \dot{y}\ddot{y}$) M$b = \frac{\dot{s}\dot{y}}{\ddot{s}}$, and the ray of curvature CM $= \frac{\dot{s}^3}{\dot{x}\ddot{y}}$. For example, if $ay = xx$, then $a\dot{y} = 2x\dot{x}$, $a\ddot{y} = 2\dot{x}^2$, $\dot{x}^2 + \dot{y}^2 = \frac{4xx+aa}{aa} \times \dot{x}^2 = \frac{4y+a}{a} \times \dot{x}^2$, and M$b$ =

$\frac{\dot{s}^2}{\ddot{y}} = 2y + \frac{1}{2}a$. If the ray of curvature be expressed by R, the variation of curvature (according to Sir *Isaac Newton's* explication) will be as $\frac{\dot{R}}{\dot{s}}$. But we have insisted on this subject at length in chap. 11, b. I.

871. Resuming the suppositions in art. 869, let (*fig.* 322) ST $= p$, then the ray of curvature at P, viz. PC $= \frac{r\dot{r}}{\dot{p}}$; and, if CI be perpendicular to SP in I, IP $= \frac{p\dot{r}}{\dot{p}}$. This was demonstrated in art. 384, and may be briefly shown thus. Let S*t* be perpendicular to *pt* the tangent at *p*, and the arks *tn*, *pu*, described from the centre S, meet ST and SP in *n* and *u*. Then the angles PC*p*, TS*t*, being equal, PC will be to ST in the ultimate ratio of P*p* to *tn*; but IP is to PC in the ultimate ratio of *pu* to P*p*; consequently IP is to ST in the ultimate ratio of *pu* to *tn*, or (because the angles SP*p*, ST*t*, are ultimately equal) of P*u* to T*n*, that is, of $\dot{r}$ to $\dot{p}$; therefore IP $= \frac{p\dot{r}}{\dot{p}}$, and PC $=$ IP $\times \frac{r}{p} = \frac{r\dot{r}}{\dot{p}}$. And by substituting for $\dot{p}$ the fluxion of $\frac{r\dot{z}}{\dot{s}}$, or (supposing the circle AD to be described with the given radius SA from the centre S, SP to meet this circle always in D, SA $= a$, AD $= c$, and consequently $\dot{z} = \frac{r\dot{c}}{a}$) of $\frac{rr\dot{c}}{a\dot{s}}$, and supposing $\dot{c}$, $\dot{z}$, $\dot{s}$, or $\dot{r}$, constant, various forms may be derived for expressing the ray of curvature CP, or IP half the chord of the circle of curvature that passes through S. To give one of the most simple examples, let $\dot{s} : \dot{z} :: a^n : r^n$, as in the figures constructed in art. 393; then $p = \frac{r\dot{z}}{\dot{s}} = \frac{r^{n+1}}{a^n}$, $\frac{\dot{p}}{p} = \overline{n+1} \times \frac{\dot{r}}{r}$, IP $= \frac{p\dot{r}}{\dot{p}} = \frac{r}{n+1}$, and PC $= \frac{1}{n+1} \times \frac{a^n}{r^{n-1}}$.

872. The

872. The rest remaining as in the last article, suppose S to be a radiating point, SP any ray incident upon the curve AP, and reflected by it so as to touch *the caustic* at m. Then the angle $CPm = CPS$; and the reflected ray Pm will be to the incident ray SP (or r) as 1 is to $\frac{2r}{p} \times \frac{\dot{p}}{\dot{r}} \mp 1$, where this unit is to be added or substracted according as the ark at P has its convexity or concavity towards the radiating point S. For if CR be perpendicular to Pm in R, PR bisected in q, and Pf be taken on the reflected ray equal to the incident ray PS; then (art. 410) $qf : qR :: qR : qm$, and Pq being equal to $\frac{p\dot{r}}{2\dot{p}}$, it follows, that $Pm : SP :: \frac{p\dot{r}}{2\dot{p}} : r \mp \frac{p\dot{r}}{2\dot{p}}$. For example, if ; : ; : : $a^n : r^n$, then $\frac{r}{p} \times \frac{\dot{p}}{\dot{r}} = n + 1$, and $Pm : SP :: 1 : 2n + 1$.

873. Suppose the curve AP (*fig.* 322, *N.* 2) to refract the ray SP, let PM be the refracted ray, and touch *the caustic* in this case at M. The rest of the construction remaining the same as before, let Cr be perpendicular to PM in r, $PR = e$, $Pr = f$; $PM = x$, and let the constant ratio of the sine of incidence to the sine of refraction (or of CR to Cr) be that of n to 1; then (by art. 413) $PM : rM :: x : x - f :: CR \times SP \times Pr : Cr \times SR \times PR :: nfr : \overline{r \mp e} \times e$; consequently $x = \frac{nffr}{nfr - er \pm ee}$, e being equal to $\frac{p\dot{r}}{\dot{p}}$ and $f = \frac{\dot{r}}{n\dot{p}} \times \sqrt{\overline{nn - 1} \times rr + pp}$.

874. Suppose the curve AP (*fig.* 322, *N.* 1) to be described by any centripetal forces, and the force that acts at any point P will be directly as the square of the velocity at P, and inversely as half the chord of the circle of curvature that is in the direction of the force: when it is directed towards a given centre S, the area described by the ray SP about S flows uniformly; the velocity at any point P is inversely as ST the perpendicular from S on the tangent, and is to the velocity by which a circle could be described about S at the same distance SP by the same cen-

tripetal force as $\sqrt{\frac{\dot{r}}{r}}$ to $\sqrt{\frac{\dot{p}}{p}}$; and the force at P is as $\frac{\dot{p}}{p^3\dot{r}}$, because the velocity is as $\frac{1}{p}$, and PI $= \frac{p\dot{r}}{\dot{p}}$. The same force is as $\frac{\ddot{s}\,\dot{s}}{-\dot{r}}$, or as $\frac{\dot{z}^2}{r} \pm \ddot{r}$, the fluxion of the area (or $r\dot{z}$) being supposed constant. Thus if $\dot{s} : \dot{z} :: a^n : r^n$, the centripetal force directed towards S will be inversely as the power of SP of the exponent $2n + 3$ (because $p = \frac{r^{n+1}}{a^n}$ and $\frac{\dot{p}}{p^3} = \overline{n+1} \times \frac{a^{2n}\dot{r}}{r^{2n+3}}$), and the velocity at P to the velocity in a circle at the same distance as $\sqrt{\frac{\dot{r}}{r}}$ to $\sqrt{\frac{\dot{p}}{p}}$, that is, as 1 to $\sqrt{\overline{n+1}}$. The demonstration that was promised in art. 451 may be deduced in the following manner.

875. Let AMB (*fig.* 323) be any figure that can be described by a centripetal force directed towards S that is always as the power of the distance SM of the exponent m. Constitute the angle ASL : ASM :: $m+3 : 2$; and, supposing SA $= 1$, SM $= x$, SL $= r$, let $r = x^{\frac{m+3}{2}}$; that is, let the angle ASL be to the angle ASM, and the logarithm of the ray SL to the logarithm of SM always in the same invariable ratio of $m+3$ to 2; then the curve ALD may be described by a centripetal force directed towards S that always varies as the power of the distance SL whose exponent is $\frac{4}{m+3} - 3$. For let SQ and SP be perpendicular to the respective tangents of AM and AL in Q and P, SQ $= y$, and SP $= p$. Then, by the supposition, $\frac{\dot{y}}{y^3\dot{x}} = ex^m$, where e represents an invariable quantity. By finding the fluents $\frac{1}{y^2} = 2K - \frac{2ex^{m+1}}{m+1}$, where K denotes an invariable

variable quantity, according to art. 735. The triangles SMQ, SLP, being similar (art. 394), it follows, that $\frac{1}{p^2} = \frac{x^2}{r^2 y^2} =$ (because $r^2 = x^{m+3}$) $\frac{1}{y^2 x^{m+1}} = \frac{2K}{x^{m+1}} - \frac{2e}{m+1} = 2Kr^{-\frac{2m+2}{m+3}} - \frac{2e}{m+1}$, and $\frac{\dot{p}}{p^3 \dot{r}} = \frac{4m+4}{m+3} \times Kr^{\frac{-3m-5}{m+3}}$, or as the power of r of the exponent $\frac{4}{m+3} - 3$. If the rays from S be perpendicular to the curve AMB in A and B, and to the curve ALD in A and D, the angle ASD : ASB : : $m + 3 : 2$, by the construction.

876 (*Fig.* 322). Suppose the centripetal force to be always the same at equal distances from the centre S. Let e and V denote the forces at the respective distances SA and SP, h and u the velocities at A and P, let SA $= a$, and SP $= r$; then $uu = \text{F.} - 2\text{V}\dot{r}$ (by art. 435); in determining which fluent, care must be taken that u become equal to h when $r = a$. When V is to e as r to a, or as aa to rr, the trajectory is a conic section, by art. 445 and 446; and when V : e : : $a^3 : r^3$, the trajectory may be constructed by the areas of conic sections, as has been already shown by several authors. When V : e : : $a^5 : r^5$, the trajectory is constructed, in some particular cases only, by the areas of conic sections (or circular arks and logarithms), but is constructed in general by the arks of conic sections. In this case a body may continually descend in a spiral line towards the centre, and yet never descend so far as to enter within a circle of a certain radius; and a body may recede for ever from the centre, so as never to arise to a certain finite altitude, but revolve in a spiral that is always within a certain circle. This remarkable circumstance could not take place in the trajectories that are described in the former cases, which have been already constructed by others; and therefore we have chosen the construction of this case for an example of the method of determining the trajectory from the law of the centripetal force.

877. Let h denote the velocity, and AG or GA be the direction of the body at any given point A. Let h be to the velocity with which the body would describe a circle at the same distance by the same centripetal force as $\sqrt{1+mm}$ to $\sqrt{2}$; that is, let (*fig.* 324) $hh : ae :: 1+mm : 2$. Let SG be perpendicular to AG in G, and any ray SP from S meet the trajectory in P, and the circle AX described from the centre S in X, SA $= a$, SG $= b$, SP $= r$, ST (the pependicular on the tangent at P) $= p$, the ark AX $= c$; and the same fluxions be represented by $\dot{s}$ and $\dot{z}$, as before. Then $uu = \text{F}$, $\frac{-2a^5e\dot{r}}{r^5} = \frac{a^5e}{2r^4} + \text{K} =$ (because when $r = a$, then $uu = hh$ $= \frac{ae}{2} + \text{K} = \frac{1+mm}{2} \times ae$, so that $\text{K} = \frac{mmae}{2}$) $\frac{a^4+mmr^4}{2r^4}$ $\times ae = hh \times \frac{bb}{pp} = \frac{1+mm}{2} \times \frac{aebb\dot{s}^2}{rr\dot{z}^2}$; consequently $\dot{s}^2 : \dot{z}^2$ $:: a^4 + mmr^4 : \overline{1+mm} \times bbr^2$, and $\dot{r}^2 : \dot{z}^2$ $(= \frac{rr\dot{c}^2}{aa}) ::$ $a^4 + mmr^4 - \overline{1+mm} \times bbr^2 : \overline{1+mm} \times bbr^2$; therefore $\dot{c} = \frac{\mp \dot{r}ab\sqrt{1+mm}}{\sqrt{a^4 - \overline{1+mm} \times bbr^2 + mmr^4}}$. The ratio of $\sqrt{1+mm}$ to 1 is that of the velocity at A in the trajectory to the velocity that would be acquired by an infinite descent to A. If $m = o$, $\dot{c} = \frac{\dot{r}ab}{\sqrt{a^4 - bbr^2}}$, and the trajectory is an ark of a circle that passes through S, described upon a diameter equal to $\frac{aa}{b}$; which is agreeable to art. 437.

878. The trajectory is constructed by circular arks and logarithms (and is of that kind of spiral lines which were mentioned at the latter end of art. 343), when the body sets out from A in the trajectory with a velocity that is to the velocity in a circle at the same distance SA as SA is to $\sqrt{\text{SA}^2 \mp \sqrt{\text{SA}^4 - \text{SG}^4}}$. In this case (supposing $\text{SA}^2 : \text{SG}^2 :: n : 1$, or $aa = nbb$), $1 + mm :$ $2 :: a^2 : a^2 \mp \sqrt{a^4 - b^4} :: n : n \mp \sqrt{nn-1}$, $m = n$

$\pm\sqrt{nn-1}$, and $\overline{1+mm} \times bb = \frac{2aa}{n \mp \sqrt{nn-1}} = 2maa$; consequently $\dot{c} = \frac{\mp \dot{r} aa\sqrt{2m}}{aa-mrr}$. Suppose, 1, that the velocity at A in the trajectory is to the velocity in a circle at the distance SA as $\sqrt{n}$ to $\sqrt{n+\sqrt{nn-1}}$ (in which case $m = n - \sqrt{nn-1}$), upon SA produced take Sk : SA : : 1 : $\sqrt{m}$, describe the circle kxK from the centre S, take the ark kK (on the same side of k that AG is of A) equal to $\frac{1}{\sqrt{2}} \times \log. \frac{1-\sqrt{m}}{1+\sqrt{m}}$, the *modulus* being S$k$, join SK, and it shall be the tangent of the trajectory at the point S. To find any other point of the trajectory, as P; let SK $= d$, take the ark K$x = \frac{1}{\sqrt{2}} \times \log. \frac{d-r}{d+r}$, join S$x$, and upon the right line Sx take SP$=r$. For, suppose the ark K$x = y$, then, by art. 731, $\dot{y} = \frac{\mp dd\dot{r}\sqrt{2}}{dd-rr} =$ (because dd : aa : : 1 : m) $\frac{\mp aa\dot{r}\sqrt{2}}{da-mrr}$ and $\dot{c} = \frac{a\dot{y}}{d} = \frac{\mp aa\dot{r}\sqrt{2m}}{aa-mrr}$, as it ought to be. Therefore describe an equilateral hyperbola Knv having its centre in S and vertex in K; let any right line Srn meet the hyperbola in n and the tangent at K in r, then let the circular sector SKx : SKn : : $\sqrt{2}$: 1, and SP be taken upon Sx equal to Kr, and P shall be a point in this trajectory. 2. Let the velocity at A in the trajectory be to the velocity in a circle at the distance SA (*fig.* 325) as $\sqrt{n}$ to $\sqrt{n-\sqrt{nn-1}}$, then $m = n + \sqrt{nn-1}$, SK is to be taken less than SA in the ratio of 1 to $\sqrt{m}$, the sector SKx : SKn : : $\sqrt{2}$: 1; and SP is to be taken upon the ray Sx, so that Kr : SK : : SK : SP.

879. In the first case [when the velocity at A (*fig.* 324) in the trajectory is to the velocity in a circle at the same distance as a to $\sqrt{a^2+\sqrt{a^4-b^4}}$], if the body set out from A with the direction GA, it will perform its revolutions in a spiral always within the circle Kxz, and never can arise to the altitude SK from the centre S; because Kr (to which the distance SP is always

 equal)

equal) cannot become equal to SK, while the area SKn or ark Kx are finite. The area described by the ray SP about the centre S is always to the hyperbolic area generated by the right line rn in the invariable ratio of $\sqrt{2}$ to 1; because $\dot{x} : \dot{y} :: r\sqrt{m} : a$, $\frac{r\dot{x}}{2} = \frac{\dot{y}\, rr\sqrt{m}}{2a} = \frac{\mp arr\dot{r}\sqrt{2m}}{2aa-2mrr}$; and the fluxion of the area Krn ($=$ SKn $-$ SKr) is $\frac{arr\dot{r}\sqrt{m}}{2aa-2mrr}$. Therefore if the body set out from A, with the direction AG, it will descend in the curve APS to the centre S in the time that, by proceeding in the tangent AG with its velocity at A, it would describe about S a triangle equal to $\sqrt{2} \times$ KRN, KR being supposed equal to SA. In this figure the area SoPxK (terminated by the curve SoP, the circular ark Kx, and right lines SK and Px) admits of a perfect quadrature, and is to the triangle SKr as $\sqrt{2}$ to 1.

880. In the second case, when the velocity at A (*fig.* 325) is to the velocity in a circle at the same distance as a to $\sqrt{a^2-\sqrt{a^4-b^4}}$, if the body set out from A with the direction AG, it will revolve in a spiral that always approaches to the circle Kx, but it never can descend to this circle; because SP $\left(= \frac{SK^2}{Kr}\right)$ cannot become equal to SK in any finite time. This spiral has an asymptote at a distance from S equal to $\frac{SA^2}{SK} \times \sqrt{2}$, because, by art. 877, $pp = \frac{1+mm}{a^4+mmr^4} \times bbr^4$, and the ultimate value of pp is $\frac{1+mm}{mm} \times bb =$ (in this case) $\frac{aa\sqrt{2}}{d}$.

881 (*Fig.* 326). In other cases, the trajectory may be constructed by hyperbolic and elliptic arks, from art. 805. If the velocity at A be to the velocity in a circle at the distance SA as $\sqrt{1-mm}$ to $\sqrt{2}$, and the direction at A be perpendicular to SA (or $a = b$), then by substituting, in art. 877, $-mm$ for mm,

$\dot{x} =$

$\dot{c} = \frac{-raa\sqrt{1-mm}}{\sqrt{a^{4}-1-mm\times aarr-mmr^{4}}} = \frac{-raa\sqrt{1-mm}}{\sqrt{aa-rr}\times\sqrt{aa+mrr}}$, which may be compared with $\frac{-bb\dot{p}}{\sqrt{aa-pp}\times\sqrt{bb+pp}}$, by supposing $b = \frac{a}{m}$ and $p=r$. The fluent of this last fluxion was found (art. 805, fig. 308) to be equal to $\frac{b}{a}\times AR + AE - EP$. Therefore, when the velocity at the distance SA is less than the velocity by which a circle would be described at the same distance in the ratio of $\sqrt{1-mm}$ to $\sqrt{2}$, the trajectory may be constructed in the following manner. Let SD : SA :: 1 : m, Sb : SA :: $\sqrt{1+mm}$: 1; describe an hyperbola AEZ having SA and SD for its two *semi-axes,* and an ellipse ARb having SA and Sb for its *semi-axes;* draw Ep a tangent to the hyperbola at any point E, and Sp a perpendicular to Ep; upon SA take SQ = Sp, and let the ordinate at Q meet the ellipse in R; then upon the circle Axz described from the centre S take the ark Ax : $\frac{1}{m}\times$ AR + AE − Ep :: $m\sqrt{1-mm}$: 1, upon the ray Sx take SP = Sp, then P shall be a point in the trajectory. In this case the velocity at A is such as could be acquired by a body descending to A from some greater distance by the same centripetal force.

882. When the velocity at the distance SA is to the velocity in a circle at the same distance as $\sqrt{1+mm}$ to $\sqrt{2}$, then $\dot{c} = \frac{\mp raa\sqrt{1+mm}}{\sqrt{aa-rr}\times\sqrt{aa-mmrr}} =$ (by supposing $pp = aa - rr$) $\frac{\pm\dot{p}aa\sqrt{1+mm}}{\sqrt{aa-pp}\times\sqrt{1-mm\times aa+mmpp}}$; and, by comparing this fluxion with that in art. 805, it appears that, when m is less than 1, we are to take SD : SA :: $\sqrt{1-mm}$: m, Sb : SA :: 1 : $\sqrt{1-mm}$, and to proceed in the construction as in the last article; only, after Sp and Ax are determined, we are now to take SP upon the ray Sx equal to $\sqrt{SA^2-Sp^2}$.

883. When

883. When m is greater than 1, then, by supposing $p = \frac{a\sqrt{rr-aa}}{r}$ and consequently $r = \frac{aa}{\sqrt{aa-pp}}$, $c = \frac{paa\sqrt{1+mm}}{\sqrt{aa-pp} \times \sqrt{\overline{mm-1} \times aa + pp}}$; therefore, in this case, we are to make SD : SA : : $\sqrt{mm-1}$: 1, Sb : SA : : m : $\sqrt{mm-1}$, to determine Sp and Ax, as in art. 881, and then we are to take SP (upon the ray Sx) equal to a third proportional to $\sqrt{SA^2-Sp^2}$ and SA. If upon Sx you take SP a third proportional to Sp and SA, P will be a point in the trajectory which is described by a centrifugal force directed from S that is inversely as the fifth power of the distance. When the direction of the body at the distance SA is oblique to the ray drawn from the centre S, the trajectories may be constructed in a similar manner.

884. If the curve FM (*fig.* 319) be described by powers directed in any manner whatsoever, and the force at any point M, resulting from the composition of these powers, act in the direction MK, and be measured by MK; let MK be resolved into the force MO in the direction of the ordinates MP ($=y$), and the force OK parallel to the base AP ($= x$); then, the time being supposed to flow uniformly, or the velocity at M being represented by the fluxion of the curve FM, the force MO will be measured by $\ddot{y}$, and the force OK by $\ddot{x}$, by art. 465 and 466; but we insisted on this, and its use, in book I. chap. 11, article 465, &c.

885. Let a body descend along the curve FPA (*fig.* 327) by its gravitation towards S, the time of the motion be represented by t, the velocity at any distance SP or r by u, the centripetal force at the same distance by g, the ark FP by s; then the motion of the body along the curve is accelerated by the force $\frac{-g\dot{r}}{\dot{s}} = \frac{\dot{u}}{\dot{t}} =$ (because $\dot{t} = \frac{\dot{s}}{u}$) $\frac{u\dot{u}}{\dot{s}}$; consequently $u\dot{u} = -g\dot{r}$, $uu =$ F. $- 2g\dot{r}$ and $\dot{t} = \frac{\dot{s}}{\sqrt{F.-2g\dot{r}}}$. When the gravity is uniform,

form, and acts in parallel lines, let z be the space described in a vertical line from the beginning of the descent, then $uu = F.2g\dot{z} = 2gz$, $\dot{t} = \frac{\dot{z}}{\sqrt{2gz}}$, and $t = \sqrt{\frac{2z}{g}}$. The gravity being still uniform, let (*fig.* 238, *N.* 1) the body begin to descend along the curve DMS from D, MN be perpendicular to the horizontal line DA in N, the ark SM $= s$, MN $= z$, and t represent the time of descent from M to the lowermost point S; then $\dot{t} = \frac{\dot{s}}{\sqrt{2gz}}$. If DMS be an ark of a semi-cycloid that has its axis perpendicular to the horizon, the diameter of the generating circle $= a$, AS $= b$, then (by the second property of this figure in art. 805) $\dot{s} : -\dot{z} :: \sqrt{a} : \sqrt{b-z}$, and $\dot{t} = \frac{-\dot{z}\sqrt{a}}{\sqrt{2gz \times \overline{b-z}}}$. If N be to 1 as the semi-circumference of a circle to the diameter, N shall represent the fluent of $\frac{-\dot{z}}{2\sqrt{z \times \overline{b-z}}}$, that is generated while z becomes equal to b; consequently the time of descent in the ark of the cycloid DMS is expressed by N $\times$ $\sqrt{\frac{2a}{g}}$, and is to the time of descent in the axis a (viz. $\sqrt{\frac{2a}{g}}$) as N to 1, as we found in art. 408.

886. But when DMS is an ark of a circle, t is a fluent of a higher kind, and is not to be represented by the areas of conic sections, but by their arks. Let C (*fig.* 238, *N.* 2) be the centre of the circle, HCS the vertical diameter, MV perpendicular to HS in V, HS $=$ E, CA $=$ F; then $\dot{s} : -\dot{z} ::$ CS : MV $:: \frac{1}{2}$E : $\sqrt{\frac{1}{4}EE - FF - 2Fz - zz}$, and $\dot{t} = \frac{-E\dot{z}}{2\sqrt{gz} \times \sqrt{\frac{1}{4}EE - FF - 2Fz - zz}}$.

Let this fluxion be compared with ($\dot{Q} =$) $\frac{-bb\dot{z}}{2\sqrt{az} \times \sqrt{bb - 2ez - zz}}$, the fluent of which was determined in art. 805; and we have $bb = \frac{1}{4}EE - FF$, or $b =$ AD, $2F = 2e = \frac{bb-aa}{a}$, and $a =$

$\frac{1}{2}$E

$\frac{1}{2}E - F = SA$. Therefore, let S be the centre, A the vertex, and SD the asymptote of the hyperbola AE; produce HD till it meet Sb perpendicular to SA in k, take Sb = Dk, and describe the ellipsis ARb; let SQ or SP = $\sqrt{ax}$; and the fluent Q will be represented by $\frac{AD}{SA} \times AR + AE - EP$, by art. 805; t the time of descent from M to S will be expressed by $Q \times \frac{HS\sqrt{SA}}{AD^2\sqrt{2g}}$, and is to the time of descent in the vertical SA as Q to $\frac{AD^2}{CS}$.

887. It follows, from art. 807, that if the semi-circumference be to the diameter as N to 1, and HA : AD : m : : 1, then the time in the whole ark DMS will be represented by $\frac{HS \times N}{\sqrt{2g \times HA}}$ $\times \overline{1 - \frac{1}{4mm} + \frac{9}{64m^4}} - \&c.$ the ultimate value of which, when SA is supposed to vanish, is $\sqrt{\frac{HS}{2g}} \times N$. Therefore the time of descent in the ark DMS is to this ultimate value of t (which is said to be the time in an evanescent ark, and, by art. 885, is equal to the time in any ark of a cycloid that has the diameter of the generating circle equal to $\frac{1}{2}$ CS) as $\frac{\sqrt{mm+1}}{m} \times$ $\overline{1 - \frac{1}{4mm} + \frac{9}{64m^4} - \frac{9}{256m^6}} + \&c.$ to 1. By the sequel of the same, art. 807, if SH : SA : : n : 1, then the whole time in the ark DMS will be expressed by $N\sqrt{\frac{E}{2g}} \times$ $\overline{1 + \frac{1}{4n} + \frac{9}{64n^2}} + \&c.$ and the time in DMS will be to the time in an infinitely small ark (or the ultimate value of t) as $1 + \frac{1}{4n} + \frac{9}{64nn} + \frac{25}{256n^3} + \&c.$ to 1. When DMS is

is a quadrant, the time of descent is measured by the arks of the *lemniscata*, of which we gave an easy construction in article 803 (*fig.* 307).

888. If a body descend or ascend in the vertical line z in a *medium*, and the resistance be represented by R, its motion is accelerated or retarded by $g \pm R = \frac{\mp\dot{u}}{\dot{t}} = \frac{\mp u\dot{u}}{\dot{z}}$; and $\mp u\dot{u} = \overline{g \pm R} \times \dot{z}$: For example, if the resistance be as the square of the velocity, and a denote the velocity when the resistance is equal to the gravity, or $R : g :: uu : aa$, then $\mp u\dot{u} = g\dot{z} \times \frac{aa \pm uu}{aa}$, $\dot{z} = \frac{aa}{g} \times \frac{\mp u\dot{u}}{aa \pm uu}$, and $\dot{t} = \frac{\dot{u}}{g \pm R} = \frac{aa}{g} \times \frac{\dot{u}}{aa \pm uu}$; whence z and t may be computed from u by logarithms or circular arks. See art. 542. When the body descends along a curve line, it is accelerated by the excess of the force $\frac{-g\dot{r}}{\dot{s}}$ above R, which is therefore equal to $\frac{u\dot{u}}{\dot{s}}$; and if it ascends, the sum of these forces is equal to $\frac{-u\dot{u}}{\dot{s}}$. When a trajectory is described in a *medium*, and the centripetal force is directed towards S (*fig.* 322), let this force at any point P be to the centripetal force at P by which the same trajectory would be described in a void as z to a, and (retaining the same symbols as in art. 869) the resistance at P will be as $\frac{\dot{z}}{pp\dot{s}}$, or, if the area of the figure be supposed to flow uniformly, as $\dot{z} : \dot{s}$ (by art. 452), and is to the centripetal force at P in the *medium* as $p\dot{r}\dot{z}$ to $2z\dot{s}\dot{p}$. If the resistance R be in the compound ratio of the density D and square of the velocity uu, then D is as $\frac{R}{uu}$, or (because uu is as $\frac{ac}{pp}$) as $\frac{\dot{z}}{az\dot{s}}$: and if the curve be such as can be described in a

void

void by a force directed towards S that is as any power of the distance, D will be inversely as $\frac{r\dot{s}}{\dot{r}}$. If the centripetal force in the medium be uniform, and act in parallel lines, and y be an ordinate in the direction of the force, then the resistance will be to the gravity as $\ddot{y}$ to $2\dot{y}^2$; and if R be as Duu, then D will be as $\frac{\ddot{y}}{\dot{y}\,\dot{s}}$.

889. Suppose FPA (*fig.* 327) to be the figure which is assumed by a chain that is perfectly flexible, and gravitates towards the given point S. Then ST the perpendicular from S on PT the tangent at P shall be inversely as F. $g\dot{r}$, and the tension of the chain at any point P inversely as ST, by art. 567. If FPA be the line of swiftest descent from F to the lowermost point A, SA $= a$, SP ($= r$) meet the circle AD described from the centre S in D, the ark AD $= c$, and u be to a as the velocity at P to the velocity acquired at A; then $\dot{c} = \frac{au\dot{r}}{r\sqrt{rr-uu}}$, by art. 581 and 582. If the gravity act in parallel lines, let PM ($= y$) be an ordinate in the direction of the force, FM $= x$, PM $= y$, the ark FP $= s$; then if FPA be the *catenaria*, $\frac{\dot{x}}{\dot{s}}$ will be as F. $g\dot{y}$, by article 568. And if FPA be the line of swiftest descent, u denote the velocity acquired at P (or $u = \sqrt{\text{F. } 2g\dot{y}}$), and a the velocity acquired at the lowermost point A, then $\dot{s} : \dot{x} :: a : u$, by art. 575 and 576.

890. The base AP (*fig.* 319) being represented by x, and the ordinate PM by y, if the F. $y\dot{x}$, be computed, and the expression be made to vanish when $x = o$, according to art. 735, it will give the area APMF. When the fluent is negative, it gives the area on the other side of PM. For example, let $y = x^m$, then F. $y\dot{x} = \text{F. } x^m\dot{x} = \frac{x^{m+1}}{m+1}$, which gives the area when m is any positive number, or is a negative number less than 1. But when m is

m is a negative number greater than unit, this expression is negative, and gives the area on the other side of PM (*fig.*322). The area generated by the ray SP about S (according to the symbols in art. 869) is the fluent of $\frac{r\dot{z}}{2}$ or of $\frac{rr\dot{c}}{2a}$. We have had many examples above of the computation of areas from those theorems. There are several general theorems for computing the area described above, as in art. 752, 819, 830, 832, &c.

891. The solid generated by the area APMF (*fig.*319) about the axis AP is found by computing F. $2Ny^2\dot{x}$, where N denotes the ratio of the semi-circumference to the diameter. For example, let the figure be any conic section, AP the axis, and the general equation of the figure being $yy = Axx + Bx + C$, the solid generated by APMF about AP will be equal to $\frac{2NAx^3}{3} + NBx^2 + 2NCx$. Let Ap be taken on the other side of A equal to AP, and pm be the ordinate at p, then $pm^2 = Axx - Bx + C$; consequently the solid generated by the area ApmF about the axis Ap will be equal to $\frac{2NAx^3}{3} - NBx^2 + 2NCx$. Therefore the solid generated by the area PMmp is equal to $\frac{4NAx^3}{3} + 4NCx$. When $x = o$, $yy = C$; consequently the cylinder generated by the rectangle PHhp (HFh being parallel to Pp) is equal to $4NCx$; and the excess of the frustum generated by the area PMmp above this cylinder is $\frac{4N}{3} \times Ax^3 =$ (supposing $Pp = 2x = v$) $\frac{NAv^3}{6}$; which (if PZ : Pp : : $\sqrt{A}$: 1) is $\frac{1}{4}$ of the cone generated by the right-angled triangle PZp about Pp, and is always of the same magnitude when v and A are the same. The frustum is greater or less than the cylinder according as A is positive or negative; and they are equal when $A = o$; that is, when the figure is a parabola. In this manner the properties of these solids described above, *p.* 24, are briefly demonstrated. When the value of F. $2Nyy\dot{x}$ is

is negative, it represents the solid that is generated by the area on the other side of the ordinate PM. Thus if $y = x^{-m}$, then $\dot{F}.\,2Nyy\dot{x} = \frac{2Nx^{-2m+1}}{-2m+1} = \frac{-2Nyyx}{2m-1}$, which expression is negative when m is greater than $\frac{1}{2}$, and represents the limit to which the solid generated by the hyperbolic area on the other side of PM continually approaches whilst that area is supposed to be produced. See art. 307, &c.

892. The ark FP is the fluent of $\dot{s}$, or of $\sqrt{\dot{x}^2+\dot{y}^2}$. For example, let $ayy = x^3$, then $y = \frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, $\dot{y} = \frac{3x^{\frac{1}{2}}\dot{x}}{2a^{\frac{1}{2}}}$, $\dot{s} = \frac{\dot{x}}{2} \times \sqrt{\frac{9x+4a}{a}}$, and by art. 727, $s = \frac{\overline{9x+4a}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}} + K$. If we suppose $s = o$ when $x = o$, then $K = \frac{-8a}{27}$ and $s = \frac{\overline{9x+4a}|^{\frac{3}{2}} - 8a^{\frac{3}{2}}}{27a^{\frac{1}{2}}}$. In like manner, if we make use of the notation in art. 869 (*fig.* 322, N. 1), $\dot{s} = \frac{r\dot{r}}{\sqrt{rr-pp}}$. Suppose, for example, $app = r^3$, then $\dot{s} = \frac{r\dot{r}\sqrt{a}}{\sqrt{arr-rrr}} = \dot{r}a^{\frac{1}{2}} \times \overline{a-r}|^{-\frac{1}{2}}$, and (art. 727) $s = 2a^{\frac{1}{2}} \times \overline{a-r}|^{\frac{1}{2}} = 2\sqrt{aa-ar}$. If we suppose AP (*fig.* 329) to be a parabola, S the focus, and A the vertex, then T will be always found in the right line AE perpendicular to SA; and the parabolic ark AP = PT + log. $\overline{\text{ST} + \text{TA}}$, the *modulus* being SA. For let SA = a, ST = p, SP = r, AP = s, and PT = q, then $pp = ar$, $\dot{s} = \frac{r\dot{r}}{\sqrt{rr-pp}} = \frac{r\dot{r}}{\sqrt{rr-ar}} =$ (because $q = \sqrt{rr-ar}$ and $\dot{q} = \frac{2r\dot{r}-a\dot{r}}{2\sqrt{rr-ar}}$) $\dot{q} + \frac{a\dot{r}}{2\sqrt{rr-ar}}$. But if u = ST + TA

$+ \mathrm{TA} = \sqrt{ra} + \sqrt{ra - aa}$, then $\frac{\dot{u}}{u} = \frac{\dot{r}}{2\sqrt{rr - ar}}$; consequently $\dot{s} = \dot{q} + \frac{a\dot{u}}{u}$, and $s = q + \log. u$, the *modulus* being equal to a. See art. 746 and 845, for the mensuration of circular arks, and art. 806, 807, 808, for hyperbolic and elliptic arks.

893. The surface generated by the ark s, when the figure revolves about the base (the ordinate being represented by y and base by x), is F. $4Ny\dot{s}$ or F. $4Ny\sqrt{\dot{x}^2 + \dot{y}^2}$, by art. 229. Thus if the parabola AP (*fig.* 329) revolve about the axis ASM, PM being perpendicular to AS in M, $\mathrm{PM} = y = 2\,\mathrm{AT} = 2\sqrt{ar - aa}$, and $\dot{s} = \dot{r}\sqrt{\frac{r}{r-a}}$; consequently $y\dot{s} = 2\dot{r}\sqrt{ar}$, the surface generated by the ark AP is $\frac{16N}{3} \times r\sqrt{ra} + \mathrm{K} = \frac{16N}{3} \times \overline{\mathrm{SP} \times \mathrm{ST} - \mathrm{SA}^2}$, and (if SE be a mean proportional betwixt SP and ST) this surface is to the circle of the radius AE as 8 to 3.

894. Let C (*fig.* 330) be the centre, CD half the transverse axis, and CA half the second axis of the ellipse ADB, F the focus, PN perpendicular to CD, PM perpendicular to CA, and PK perpendicular to the curve meet CD in K, $\mathrm{CA} = a$, $\mathrm{CD} = b$, $\mathrm{CF} = c$, $\mathrm{CN} = x$, $\mathrm{PN} = y$, and the ark $\mathrm{AP} = s$; then $\mathrm{NK} : \mathrm{NC} :: a^2 : b^2$, or $\mathrm{NK} = \frac{a^2x}{b^2}$, $\mathrm{PN}^2 = \frac{aa}{bb} \times \overline{bb - xx}$, $\mathrm{PK}^2 = a^2 - \frac{a^2c^2x^2}{b^4}$, and $\mathrm{PK} = \frac{a}{bb} \times \sqrt{b^4 - c^2x^2}$. But $\dot{s} : \dot{x} :: \mathrm{PK} : \mathrm{PN} = y$, $y\dot{s} = \frac{a\dot{x}\sqrt{b^4 - c^4x^2}}{bb} =$ (supposing $c ; b :: b : d = \mathrm{CG}$) $\frac{ac\dot{x}\sqrt{dd - xx}}{bb}$. Therefore let CA and NP meet the circle GZE described from the centre C in E and Z, and when the figure is supposed to revolve about the axis CD, the surface generated by the elliptic ark AP will be to the area CEZN as $4N \times ac$ to bb; and if DI perpendicular to CD meet GZE in I, the whole surface of the spheroid

will be to the surface of the sphere of the radius CA as $\frac{4Nac}{bb} \times$ CEID to $4Naa$, that is, as EI + CA to 2CA. In like manner, if PK produced meet AC in k, Mk : MC ($= y$) :: $b^2 : a^2$, and P$k = \frac{b}{aa} \times \sqrt{a^4 + c^2y^2}$; let DP $= \dot{s}$, and $\dot{s} : \dot{y}$ :: Pk : PM, or PM $\times \dot{s} = \dot{y} \times$ Pk; consequently the fluxion of the surface generated by the ark DP about the axis CA is $\frac{4Nb\dot{y}}{aa} \times \sqrt{a^4 + c^2y^2}$ $=$ (if $c : a :: a : e =$ Cg) $\frac{4Nbc}{aa} \times \dot{y}\sqrt{ee+yy}$, the fluent of which is $\frac{2Nb}{e} \times y\sqrt{ee+yy} + 2Nb \times \log.\overline{y+\sqrt{ee+yy}}$ (the *modulus* being equal to e or Cg) $= 2\mathrm{N} \times \mathrm{CM} \times \mathrm{P}k + 2\mathrm{N}b \times$ log. $\overline{\mathrm{CM} + \mathrm{P}k} \times \frac{\mathrm{C}g}{\mathrm{CD}}$. Hence the surface generated by the elliptic quadrant DPA about the axis CA is $2Nb \times \overline{b + \log.\, a \times \frac{b+e}{c}}$; and the surface of this spheroid is to the surface of a sphere of the radius CD as CD + log. $\frac{\mathrm{C}g \times \mathrm{CA}}{\mathrm{DF}}$ to 2CD, the *modulus* being Cg. These constructions agree with Mr. *Cotes's Harmon. Mensurar. p.* 28 and 29, where he illustrates the transition from circular arks to logarithms (or from the measures of angles to the measures of ratios), that so often occurs in the resolution of the various cases of a problem, from an analogous transition observed long ago by *Vieta* in the resolution of cubic equations; the roots of which are in some cases obtained by trisecting an ark, and in other cases by what may be called the trisecting a ratio (*i. e.* interposing two mean proportionals betwixt the terms of the ratio); so that the trigonometrical and logarithmical canon are mutually supplements to each other. The harmony of those measures, which was so much considered by this excellent author, may be further illustrated by the resolution of the two following useful problems relating to the spheroid.

895. In plain sailing the meridians are supposed parallel, and the degrees of longitude as well as those of latitude are supposed equal; whereas the meridians intersect each other in the pole, the degrees of longitude decrease in the same proportion as the semi-diameters of the parallels of latitude, and the degrees of latitude (because of the oblate figure of the earth) increase from the equator towards the poles. In order to correct some of the errors that arise in Navigation from these false suppositions, a projection was invented (commonly called *Mercator's Chart*) in which the meridians are still supposed parallel, and the degrees of longitude enlarged as in the former, but the degrees of latitude upon the meridians are enlarged in the same proportion. The arks of the meridian thus enlarged (or the *meridional parts*) are found in a sphere or spheroid by the following theorems. Let the ark DH (*fig.* 331, *N.* 1), or angle DCH, be the latitude for which the meridional parts z are required, HE its sine, let CT bisect the ark Hd (the complement of HD), and meet the tangent at d in T. Then, 1, in the sphere $z = \log. \frac{CD}{dT}$, the *modulus* being CD. 2. In the oblate spheroid, let Dh be an ark whose sine eh is to EH as CF the distance of the focus from the centre to CD the semi-diameter of the equator; let Ct bisect the ark dh, and meet dT in t; then $z = \log. \frac{CD}{dT} - \frac{CF}{CD} \times \log. \frac{CD}{dt}$. 3. In the oblong spheroid, let Dq (*fig.* 331, *N.* 2) be the ark whose tangent is to EH the sine of DH as CF to CD, and $z = \log. \frac{CD}{dT} + \frac{CF}{CD} \times Dq$.

896. For, supposing ADB to be a meridian section through the poles A and B, as in art. 894, let CA $= a$, CD $= b$, CF $= c$, CM $= y$, EH $= u$, and the elliptic ark DP $= s$. Then, to find the meridional parts z, we are to suppose the element or fluxion of the ark DP to be always enlarged in the ratio of CD the radius of the equator to PM the radius of the parallel of P, that is, $\dot{z} = \dot{s} \times \frac{CD}{PM} =$ (because $\dot{s} : \dot{y} :: PK : NK ::$

CH : CE) $\frac{b\dot{y}}{\sqrt{bb-uu}} \times \frac{CD}{PM}$ =(because PM : NK : : $bb : aa$, and NK : CE : : PN : EH) $\frac{\dot{y}}{y} \times \frac{aau}{bb-uu}$. By what we found in art. 894, Pk $= \frac{b}{aa} \times \sqrt{a^4+ccyy}$ or $\frac{b}{aa} \times \sqrt{a^4-ccyy}$, according as CD is greater or less than CA; consequently Pk being to Mk as CH to EH, we have $u = \frac{bby}{\sqrt{a^4 \mp ccyy}}$ or $y = \frac{aau}{\sqrt{b^4 \pm ccuu}}$, and (by art. 728) $\frac{\dot{y}}{y} = \frac{\dot{u}}{u} \mp \frac{ccu\dot{u}}{b^4 \mp ccuu} = \frac{\dot{u}}{u} \times \frac{b^4}{b^4 \pm ccuu}$. Therefore $\dot{z} = \frac{aab^4\dot{u}}{bb-uu \times b^4 \pm ccuu}$, that is, $\dot{z} = \frac{bb\dot{u}}{bb-uu}$ in the sphere, $\dot{z} = \frac{bb\dot{u}}{bb-uu} - \frac{b^2cc\dot{u}}{b^4-ccuu}$ in the oblate spheroid, and $\dot{z} = \frac{bb\dot{u}}{bb-uu} + \frac{b^2cc\dot{u}}{b^4+ccuu}$ in the oblong spheroid. Suppose now db to be the diameter of the circle dDb, join dH and Hb, then the triangles TdC and dHb will be similar, dT : dC : : dH : Hb : : $\sqrt{Cd-EH}$: $\sqrt{Cd+EH}$, dT $= b \times \sqrt{\frac{b-u}{b+u}}$, and the *modulus* being b, the fluxion of log. $\frac{CD}{dT}$ (or of log. $\sqrt{\frac{b+u}{b-u}}$) shall be $\frac{bb\dot{u}}{bb-uu}$. In like manner, because $ch = \frac{cu}{b}$, $dt = b \times \sqrt{\frac{bb-cu}{bb+cu}}$, the fluxion of log. $\frac{CD}{dt}$ (or of log. $\sqrt{\frac{bb+cu}{bb-cu}}$) is $\frac{b^3c\dot{u}}{b^4-ccuu}$. Therefore in the sphere z $=$ log. $\frac{CD}{Td}$ = log. CD — log. dT; and in the oblate spheroid, z = log. $\frac{CD}{dT} - \frac{CF}{CD} \times$ log. $\frac{CD}{dt}$. In the oblong spheroid,

roid, Dq is the fluent of $\frac{b^3 c\dot{u}}{b^4 + ccuu}$; consequently $z = \pm$ log. $\frac{CD}{dT} + \frac{CF}{CD} \times Dq$.

897. These logarithms are hyperbolic, or of *Napier*'s first sort; but it is easy to adapt the theorems to the tabular logarithms, and to express the meridional parts in minutes, as is usual. Thus in the sphere substract the logarithmic tangent of half the complement of the latitude from the logarithm of the radius (or 10.000000), and multiply the remainder by 7915.704467897 &c. (*viz.* the number of minutes contained in the radius divided by the *modulus* of the tables), then the product shall give the meridional parts in minutes.

898. In the oblate spheroid we have this easy rule: let CF : CD : : 1 : n, and u be the sine of the given latitude DCH for which the meridional parts are required. The table for the meridional parts being already computed in the sphere, find the meridional parts in this table for the latitude whose sine is $\frac{1}{n} \times u$, divide these by u, substract the quotient from the meridional parts in the same table for the given latitude DCH; and the remainder shall be the meridional parts for the same latitude in the oblate spheroid. This problem is resolved by infinite series, and a table of the meridional parts is computed for the oblate spheroid wherein $cc : bb :: 22 : 1000$, in an ingenious treatise published lately by the Reverend Mr. *Murdoch*, whose table may be examined by this rule; and it may likewise serve for facilitating the computation, when a different ratio is assumed for that of cc to bb. The greatest difference betwixt the meridional parts in the oblate spheroid and sphere is easily computed by finding the meridional parts in the sphere for the latitude whose sine is to the radius as 1 to n, and dividing these by n.

899. In the oblong spheroid, to find the meridional parts for the latitude whose sine is u, add to the meridional parts in the sphere for the same latitude $\frac{1}{n} \times Dq$, Dq being the ark whose tangent is $\frac{1}{n} \times u$.

900. Let PMoux (*fig.* 332) be a pyramid of an uniform density upon the rectangular base Moux, and suppose that the attraction of its particles is inversely as the power of the distance of any exponent n less than 3. Let the attraction at the given distance PC ($=a$) be represented by e, the attraction at any distance PM ($=r$) by V; then if the angles M$\mathrm{P}o$, M$\mathrm{P}x$, be continually diminished, the gravity at P towards the pyramid PMoux will be ultimately as $\frac{V\dot{r}}{3-n} \times Mo \times Mx$. For if N$nkl$ be a section of the pyramid parallel to Moux at a distance PN $=$ PC, the gravity at P towards the pyramid shall be ultimately equal to the fluent of $V\dot{r} \times Mo \times Mx$, or (because $Nl : Mx :: Nn : Mo :: a : r$ and $V : e :: a^n : r^n$) of $\frac{a^{n-2}e\dot{r}}{r^{n-2}} \times Nl \times Nn$, that is, to

$$\frac{a^{n-2}er^{3-n}}{3-n} \times Nl \times Nn = \frac{Vr}{3-n} \times Mo \times Mx.$$

901. Hence it will appear (by proceeding as in art. 642), that if a portion of a solid contained by planes that intersect each other in PH attract a particle at P, PMA be one of these planes, the right line PM meet the circle BNC described from the centre P with the given distance PC in N, MQ be perpendicular to PC and NR to PH, PC $= a$, PM $= r$, PQ $= z$, PR $= x$, the sine of the inclination of the planes to the radius as f to a; and, supposing this angle to be diminished continually till it vanish, the ultimate value of the gravity at P towards the slice of the solid contained by the planes in the direction PC be represented by q, then q will be equal to the fluent of $\frac{fVr^2x\dot{z}}{\overline{3-n} \times aa}$ or of $\frac{a^{n-2}efx\dot{z}}{3-n} \times r^{2-n}$. If PC coincide with PH, then $x : a :: z : r$, and the gravity at P will be as the fluent of $\frac{a^{n-3}ef}{3-n} \times r^{3-n}x\dot{x}$. If PH be perpendicular to PC, the gravity at P towards the portion of the solid will be ultimately

mately as the fluent of $\frac{x^{n-3}ef}{3-n} \times r^{3-n}\dot{x}\sqrt{aa-xx}$, because in this case $r : z :: a : \sqrt{aa+xx}$.

902. Suppose a solid to be generated by the figure PMA (*fig.* 332, *N.* 2) revolving about the axis PC, and the gravity at P towards this solid in the direction PC to be measured by Q, then Q $=$ $\frac{4Na^{n-2}e}{3-n} \times \mathrm{F}.\, r^{3-n}x\dot{x}$, N being supposed to denote the ratio of the semi-circumference to the diameter, as formerly. For example, if PMA be a semicircle of the radius PC, then PM $=$ 2PR, or $r = 2x$; and $Q = \mathrm{F}.\, \frac{Nea^{n-2}\, r^{4-n}\dot{r}}{3-n} =$ (when r becomes equal to $2a$) $\frac{2^{5-n} \times Na^{3}e}{3-n \times 5-n}$. If $n = 2$, $Q = \frac{8Na^{3}e}{3}$, and the gravity is the same as if the whole matter in the sphere attracted from the centre C, because the solid content of the sphere is $\frac{8Na^{3}}{3}$; and in other cases the gravity is to this attraction as $3 \times 2^{2-n}$ to $\overline{3-n} \times \overline{5-n}$. If $n = -1$, these are equal. If $n = o$, their ratio is that of 4 to 5; and if $n = 1$, the ratio is that of 3 to 4. In different spheres, when n is given, the gravity is as PC^{3-n}, because e is as PC^{-n}.

903. To find the attraction at the pole A (*fig.* 333) towards the spheroid generated by the semi-ellipsis ADB about the axis AB, suppose P to coincide with A, AC $= a$, CD $= b$, CF (F being the focus) $= c$, AR $= x$, AQ $= z$, AM $= r$; then $\mathrm{AR}^2 :: \mathrm{NR}^2 :: \mathrm{AQ}^2 : \mathrm{QM}^2$, or $xx : aa - xx :: zz : \frac{bb}{aa} \times \overline{2az - zz} :: z : \frac{bb}{aa} \times \overline{2a-z}$; consequently $z = \frac{2b^2 aax}{a^4 \mp c^2x^2}$, and (because $x : a :: z : r$) $r = \frac{2ab^2a^2x}{a^4 \mp c^2x^2}$, or $x = \frac{aa}{ccr} \times \overline{\mp bb \pm \sqrt{b^4 - ccrr}}$. Therefore $Q = \frac{2^{5-n} Na^{4-n} b^{6-2n} e}{3-n} \times$

$F.\frac{x^{4-n}\dot{x}}{\overline{a^4\mp c^2x^2}|^{3-n}}$; or, if we suppose $bb = dc$, Q will be equal to $\frac{4Na^{n+2}b^2e}{\overline{3-n}.x\,d^4} \times F.\frac{2d^2\dot{c}r\mp cr^2\dot{r}}{r^n\sqrt{dd-rr}} - F.\frac{2b^2\dot{r}}{r^n}$, which fluents are easily measured by the areas of conic sections, when n is any integer number. The upper signs are for the oblong spheroid.

904. To find the attraction at the point D (*fig.* 334) in the equator of the spheroid, let P coincide with D, DBE be a section of the solid perpendicular to its equator, PH or DH a tangent at D, HNc a circle described from the centre D with the radius Dc ($=$ CA) meet DM in N, MQ perpendicular to DE, and NR to DH, CA $= a$, CD $= b$, CF $= c$, as formerly, and DQ $= z$, DM $= r$, DR $= x$. Then $NR^2 : DR^2 :: DQ^2 : QM^2$, that is, $aa-xx : xx :: zz : \frac{aa}{bb} \times \overline{2bz-zz} :: z : \frac{aa}{bb} \times \overline{2b-z}$ and $z = 2baa \times \frac{aa-xx}{a^4\mp c^2x^2}$; consequently $r \left(= \frac{az}{\sqrt{aa-xx}}\right) = \frac{2ba^3\sqrt{aa-xx}}{a^4\mp ccxx}$.

Therefore $q = \frac{a^{n-3}ef}{3-n} \times F.\, r^{3-n}\dot{x}\,\sqrt{aa-xx} = F.\frac{8a^6b^3fe\dot{x}}{\overline{3-n}\times\overline{2aab}|^n} \times \frac{\overline{aa-xx}|^{\frac{4-n}{2}}}{\overline{a^4\mp ccxx}|^{3-n}}$, which gives the ultimate value of the gravity at D towards a slice of the spheroid contained by two planes perpendicular to its equator that intersect each other in DH, when the angle contained by the planes vanishes, by art. 901. If we suppose $c = o$ or $a = b$, the last fluxion becomes equal to $\frac{8a^{n-3}ef\dot{x}}{\overline{3-n}\times 2^n} \times \overline{aa-xx}|^{\frac{4-n}{2}} =$ (supposing $yy = aa - xx$) $\frac{a^{n-3}ef}{\overline{3-n}\times 2^n} \times \frac{-y^{5-n}\dot{y}}{\sqrt{aa-yy}}$, the fluent of which gives the ultimate value of the gravity at D towards the slice of the sphere (described upon the diameter of the equator

of

of the spheroid) that is contained by the same planes. Because the sections of the spheroid by planes perpendicular to the equator are ellipses similar to the meridian section and to one another, and the sections of the sphere by these planes are circles, the gravity at D towards the spheroid is to the gravity at D towards the sphere described upon the diameter of the equator as the former to the latter fluent, that is (supposing $cc : aa :: m : 1$), as $F.\, a^{3-n} b^{3-n} \dot{x} \times \dfrac{\overline{aa-xx}|^{\frac{4-n}{2}}}{\overline{aa\mp mxx}|^{\frac{3-n}{}}}$ to $F.\, \dot{x} \times \overline{aa-xx}|^{\frac{4-n}{2}}$. These fluxionary expressions are rational when n is an even number; and when n is an odd number they are transformed into rational expressions by supposing $x = \dfrac{az}{\sqrt{aa+zz}}$. Hence, therefore, the gravity at the equator, as well as the gravity at the poles, is measured by circular arks or logarithms when n is any integer number less than $+3$.

905. When $n=2$, the gravity at the pole or equator is easily computed from the first theorem in art. 901, viz. $q = F.\, \dfrac{a^{n-2}efzx}{3-n} \times r^{2-n} =$ (when $n=2$) $F.\, efz\dot{x}$. For when the particle P, whose gravity is required, is at A, as in art. 903, z (or AQ, supposing $AR = x$) $= \dfrac{2bb}{a} \times \dfrac{xx}{aa\mp mxx}$ and $q = \dfrac{2b^2ef}{a} \times F.\, \dfrac{x^2\dot{x}}{aa\mp mxx}$; consequently the gravity at the pole A towards the spheroid is to the gravity at A towards the sphere of the diameter AB as $bb \times F.\, \dfrac{x^2\dot{x}}{aa\mp mxx}$ to $\frac{1}{3}aa$. When the particle P is at D on the circumference of the equator, suppose, as in art. 904, $DR = x$, then $DQ = z = 2b \times \dfrac{aa-xx}{aa\mp mxx}$, and $q = F.\, efz\dot{x} = 2bef \times F.\, \dot{x} \times \dfrac{aa-xx}{aa\mp mxx}$; consequently the gravity at D on the circumference of the equator towards the spheroid

spheroid is to the gravity at D towards the sphere upon the diameter DE as F. $ab\dot{x} \times \frac{aa-xx}{aa+mxx}$ to F. $\dot{x} \times \overline{aa-xx}$ = (when $x = a$) $\frac{2a^3}{3}$; and these fluents give the same constructions by circular arks and logarithms that were described in art. 646 and 647. The gravity at any point P on the surface of the spheroid in the direction parallel to the axis, or perpendicular to it, may be computed in like manner from the theorem $q =$ F. $efz\dot{x}$; but this case is reduced to the former by art. 634. When the density varies, but so as to be uniform over any surface similar and concentric to ADBE, the gravity at any place in the plane of the equator, or axis of the spheroid, may be computed by art. 668, &c. The reader will find this subject treated in a different manner in a late ingenious essay, *Phil. Trans.* N. 449, by Mr. *Clairaut*. It was demonstrated in art. 636, &c. that if the density of the earth was uniform, its figure would be such a spheroid as is generated by an ellipsis revolving about its second axis, according to the theory of gravity; but this was assumed as an hypothesis in art. 679, 681, &c. where the density was supposed variable.

906 (*Fig.* 335). The centres of gravity and oscillation of figures are determined from art. 509 and 534. Let G be the centre of gravity, and O be the centre of oscillation of the plane figure F*fm*M when it revolves about the axis F*f*, OGA perpendicular to F*f* in A and to M*m* in P, AP $= x$, M*m* $= y$, GA $= z$, OA $= u$; then $z = \frac{F.yx\dot{x}}{F.y\dot{x}}$ and $u = \frac{F.yx^2\dot{x}}{F.yx\dot{x}}$. Thus, if $\frac{1}{2}y = x^m$, $z = \frac{F.x^{m+1}\dot{x}}{F.x^m\dot{x}} = \frac{m+1}{m+2} \times x$, or GA : PA :: $m+1 : m+2$; and $u = \frac{F.x^{m+2}\dot{x}}{F.x^{m+1}\dot{x}} = \frac{m+2}{m+3} \times x$, or OA : PA :: $m+2 : m+3$ (*fig.* 239). The centre of oscillation of a sphere was determined, in art. 536, from the

fluent

fluent of $2ny^2\dot{y} \times \overline{aa - yy}$ [supposing, in fig. 239 (*fig.* 239) the radius GE $= a$, GN $=$ PM $= y$, OG $= z$ and n to 1 as the circumference of the circle to its radius], which is $2ny^3 \times \overline{\frac{aa}{3} - \frac{yy}{5}}$; and this fluent becomes equal to $\frac{4na^5}{15}$ when PM $=$ GE or $y = a$, which being divided by $\frac{2n}{3} \times a^3 \times z$ (the solid content of the sphere multiplied by the distance of its centre of gravity from the axis of oscillation) gives $\frac{2}{5} \times \frac{aa}{z} = u$. The centre of percussion is in a right line perpendicular to AO at O. Several principles concerning the centre of gravity and its motion, that are of use in the resolution of problems, were explained in art. 511, 526, 533, 544, 551, &c. The motion of a fluid issuing from a cylindric vessel was considered in art. 537, 540, 541, &c. and an example of the method by which the principle concerning the equality of the ascent and descent of the centre of gravity is applied to this enquiry (*Comment. Petropol. tom.* 2) is described in art. 544. But the same theory has been since prosecuted more fully by the learned author, and illustrated by various experiments, in a particular treatise, entitled *Hydrodynamica.*

907. In any engine the proportion of the power to the weight, when they balance each other, is found by supposing the engine to move, and reducing their velocities to the respective directions in which they act; for the inverse ratio of those velocities is that of the power to the weight, according to the general principle of mechanics. But it is of use to determine likewise the proportion they ought to bear to each other, that when the power prevails, and the engine is in motion, it may produce the greatest effect in a given time. When the power prevails, the weight moves at first with an accelerated motion; and when the velocity of the power is invariable, its action upon the weight decreases while the velocity of the weight increases. Thus the action of a stream of water or air upon a wheel is to be estimated from the excess of the velocity of the

fluid

fluid above the velocity of the part of the engine which it strikes, or their relative velocity, only. The motion of the engine ceases to be accelerated when this relative velocity is so far diminished that the action of the power becomes equal to the resistance of the engine arising from the gravity of the matter that is elevated by it, and from friction; for when these balance each other, the engine proceeds with the uniform motion it has acquired. Let a denote the velocity of the stream, u the velocity of the part of the engine which it strikes when the motion of the machine is uniform, and $a-u$ will represent their relative velocity. Let A represent the weight which would balance the force of the stream when its velocity is a, and p the weight which would balance the force of the same stream if its velocity was only $a-u$; then $p : A :: \overline{a-u}^2 : a^2$, or $p = \frac{A \times \overline{a-u}^2}{aa}$; and p shall represent the action of the stream upon the wheel. If we abstract from friction, and have regard to the quantity of the weight only, let it be equal to qA (or be to A as q to 1); and, because the motion of the machine is supposed uniform, $p = q \times A = \frac{A \times \overline{a-u}^2}{aa}$, or $q = \frac{\overline{a-u}^2}{aa}$. The *momentum* of this weight is $qAu = \frac{Au \times \overline{a-u}^2}{aa}$, which is a *maximum* when the fluxion of $\frac{u \times \overline{a-u}^2}{aa}$ vanishes, that is, when $\dot{u} \times \overline{a-u}^2 - 2\dot{u}u \times \overline{a-u} = o$, or $a - 3u = o$. Therefore, in this case, the machine will have the greatest effect if $u = \frac{a}{3}$, or the weight $qA = \frac{A \times \overline{a-u}^2}{aa} = \frac{4A}{9}$; that is, if the weight that is raised by the engine be less than the weight which would balance the power in the proportion of 4 to 9; and the *momentum* of the weight is $\frac{4Aa}{27}$.

908. If

908. If we would likewise consider the friction arising from the motion of the weight, let 1 be to n as the weight is to the resistance of the engine which would arise from this friction, if the motion of the engine was such that the part of it impelled by the stream moved with the given velocity a; then, supposing the friction to be always in the compound ratio of the weight and velocity, the resistance of the engine arising from the same cause when the part of the wheel impelled by the stream moves with the velocity u will be $\frac{nqAu}{a}$. Suppose, therefore, $p = qA + \frac{nqAu}{a} = \frac{A \times \overline{a-u}^2}{aa}$, then $qA = \frac{A}{a} \times \frac{\overline{a-u}^2}{a+nu}$, and the *momentum* of the weight $qAu = \frac{Au}{a} \times \frac{\overline{a-u}^2}{a+nu}$; the fluxion of which being supposed to vanish, we shall find $aa - 3au - 2nuu = o$, or $u = \frac{2a}{3+\sqrt{9+8n}}$, and the weight $qA = 4A \times \frac{1+\sqrt{9+8n}}{\overline{3+\sqrt{9+8n}}^2}$; that is, the machine will have the greatest effect (according to this supposition) when $u : a :: 2 : 3 + \sqrt{9+8n}$, and the weight is to that which would balance the power as $2 + 2\sqrt{9+8n}$ to $9 + 4n + 3\sqrt{9+8n}$. For example, if $n = \frac{7}{8}$, then $u = \frac{2a}{7}$, and $qA : A :: 20 : 49$; consequently, though the velocity u be less than in the former case in the ratio of 6 to 7 (and therefore the action of the power on the wheel be greater), yet the weight that is raised is less in the ratio of 45 to 49, and the effect of the engine is less in the ratio of 270 to 343. If n be very small in respect of 1, then $u : a :: 1 : 3 + \frac{2n}{3}$, and $qA : A :: 4 + \frac{4n}{3} : 9 + 4n$ nearly. But if we would have likewise regard to the friction arising from the motion of the parts of the engine, as well as to that which arises from the elevation of the weight, the computation will be somewhat

what different. Let the friction be equal to mA when the machine moves without any charge in such a manner that the velocity of the part impelled by the stream is equal to a; and the friction will be equal to $\frac{mAu}{a}$ when this velocity is u, where we suppose m invariable, because the machine remains the same. When the motion of the engine is uniform, $p = qA + \frac{nqAu}{a} + \frac{mAu}{a} = \frac{A \times \overline{a-u}^2}{au}$, and, supposing the *momentum* of qA to be a *maximum*, u will be found by resolving the equation $u^3 + \overline{\frac{3}{2n} - 1 - \frac{m}{2}} \times au^2 - \frac{2+m}{n} \times aau + \frac{a^3}{2n}$. For example, if $n = \frac{1}{10}$ and $m = \frac{1}{10}$, u is nearly $\frac{3a}{10}$, qA is about $\frac{39A}{103}$, and the effect of the engine about $\frac{1}{9}$ of Aa or $\frac{3}{4}$ths of what it would have been if there was no friction, and u was equal to $\frac{a}{3}$.

909. Suppose that the given weight P (*fig*.336) descending by its gravity in the vertical line raises a given weight W by the line PMW (that passes over the pully M) along the inclined plane BD, the height of which BA is given; and let the position of the plane BD be required, along which W will be raised in the least time from the horizontal line AD to B. Let $AB = a$, $BD = x$, $t =$ time in which W describes DB; the force which accelerates the motion of W is $P - \frac{aW}{x}$, tt is as $\frac{xx}{Px - aW}$, and if we suppose the fluxion of this quantity to vanish, we shall find $x = \frac{2aW}{P}$ or $P = \frac{2aW}{x}$; consequently the plane BD required is that upon which a weight equal to 2W would be sustained by P; or if BC be the plane upon which W would sustain P, then $BD = 2BC$. But if the position of the plane BD be given, and W being supposed variable, it be required to find the ratio of W to P when the greatest *momentum* is produced

duced in W along the given plane BD; in this case W ought to be to P as BD to BA + $\sqrt{BD+BA} \times \sqrt{BA}$.

910. The radius CA (*fig.* 337) and angle ACB being given, let E be any point upon the ark AB, EM the sine of the angle ECA, EN the sine of the angle ECB, n any positive number, and let it be required to determine the point E when $EM^n \times EN$ is a *maximum*. Upon AC produced beyond C take CD : CA :: $n-1$: $n+1$; draw DG parallel to CB, meeting the circle AB in G, and if CE bisect the angle ACG, it will meet the circle in the point E required. For, let ER parallel to CB meet CA in R, CR $= x$, ER $= y$, and when $y^n x$ is a *maximum* (or when its fluxion vanishes), $\frac{n\dot{y}}{y} + \frac{\dot{x}}{x} = o$, by art. 728, or $n\dot{x} = \frac{\dot{y}x}{y}$. Let the tangent at E meet CA in T, CB in Z, and CG in Q, and AP perpendicular to CE meet CB in K and the circle again in H; then RT $= \frac{-y\dot{x}}{\dot{y}} = nx$, or RT : CR :: n : 1 :: ET : EZ :: AP : PK :: PH : PK; consequently HK : KA :: $n-1$: $n+1$ :: CD : CA, and DH is parallel to CB. Therefore H coincides with G, and the ark GA is bisected in E when $ER^n \times CR$ is a *maximum*, or (because ER is to EM and CR to EN in the same invariable ratio of the radius to the sine of the given angle ACB) when $EM^n \times EN$ is a *maximum*.

911. Let a fluid that moves with the velocity and direction AC strike the plane CE; and suppose that this plane moves parallel to itself in the direction CB. Take CD : CA :: 1 : 3, draw DG parallel to CB meeting the circle AB in G; and if the plane CE bisect the angle ACG, then the effect of the fluid upon CE will be greatest at the beginning of the motion (*fig.* 337, *N.* 2). But if the plane CE has already acquired a motion in the direction CB, let its velocity in this direction be to the velocity of the stream as *Aa* to AC, and let *Aa* be parallel to CB; let a circle described from the centre C with the distance C*a* meet DG in *g*; and the effect of the stream upon the plane CE will be greatest in

in this case when the plane bisects the angle aCg. For let AP and ap be perpendicular to CE in P and p, and ah perpendicular to AP in h; then the motion of the particles of the fluid in the direction perpendicular to CE will be represented by AP, their motion in the direction parallel to the plane CE by CP, the motion of the plane in the former direction by Ah, and its motion in the latter direction by ah. The action of the fluid on the plane depends on their relative velocity only, that is, on the difference of the motions AP and Ah (which is equal to $h\mathrm{P} = ap$), and on the sum or difference of the motions PC and ha, which is equal to pC. It follows, that the action of the stream on the plane CE is the same in this case as when the plane is at rest, and the stream strikes it with the direction and force aC. Let this force aC be resolved into ap perpendicular to the plane CE, and pC parallel to it; and because the latter has no effect upon the plane CE, let the force ap be resolved into the force ak parallel to CB, and pk perpendicular to it; then because the force pk has no effect to impel the plane CE in the direction CB, ak will measure the force with which any particle of the fluid impels CE in the direction CB; and the number of particles incident upon the plane CE in the same time being as ap, the effect of the stream to move the plane CE in the direction CB shall be measured by $ak \times ap =$ (Em and EN being perpendicular to Ca and CB in m and N, and consequently $ak : ap :: \mathrm{EN} : \mathrm{CE}$) $\frac{ap \times ap \times \mathrm{EN}}{\mathrm{CE}} = \mathrm{E}m^2 \times \mathrm{EN} \times \frac{\mathrm{C}a^2}{\mathrm{CE}^3}$, which is a *maximum* when CE bisects the angle aCg, by the last article; because CE and Ca are supposed to be given, $n = 2$, and $\mathrm{CD} : \mathrm{CA} :: n - 1 : n + 1 :: 1 : 3$. If A$a = o$, g coincides with G, and the stream has the greatest effect when CE bisects the angle ACG.

912. Let CV (*fig.* 338) be perpendicular to Aa in V, and CE produced meet Aa in t; take $\mathrm{VL} = \mathrm{VC} \times \sqrt{2}$, $\mathrm{V}f = \frac{3\mathrm{V}a}{2}$, join L$f$, and V$t$ the tangent of the angle VCE (VC being radius) shall be equal to $\mathrm{L}f + \mathrm{V}f$, when the plane CE is in the most advantageous position, CA the velocity and direction of the stream, and Aa the velocity and direction of the plane CE being given. For let

let pq perpendicular to CV in q meet Ca in u, let aC and VC produced meet Dg in z and d; then, because $a\mathrm{C} = 3\mathrm{C}u$, $at = 3pu =$ (because $ag = 2ap$) $\frac{3gz}{2} = \frac{3gd \mp 3dz}{2} = \frac{3gd}{2} \mp \frac{\mathrm{V}a}{2} =$ (because $gd^2 = \mathrm{C}g^2 - \mathrm{C}d^2 = \mathrm{C}a^2 - \frac{\mathrm{CV}^2}{9} = \frac{8\mathrm{CV}^2}{9} + \mathrm{V}a^2 = \frac{4}{9} \times \overline{2\mathrm{CV}^2 + \frac{9\mathrm{V}a^2}{4}} = \frac{4}{9} \times \overline{\mathrm{VL}^2 + \mathrm{V}f^2} = \frac{4}{9} \times \mathrm{L}f^2$, and $\frac{3gd}{2} = \mathrm{L}f$) $\mathrm{L}f \mp \frac{\mathrm{V}a}{2}$; and $\mathrm{V}t = at \mp \mathrm{V}a = \mathrm{L}f \mp \mathrm{V}f$. If $\mathrm{CV} = a$, $\mathrm{V}a = c$, then $\mathrm{V}t = \sqrt{2aa + \frac{9cc}{4}} \mp \frac{3c}{2}$. The negative sign is to take place when aCB is greater than a right angle.

913. When the angle ACB is right, A (*fig.* 338, *N.* 2) coincides with V, DG is perpendicular to AD, and $\mathrm{A}t = \mathrm{L}f + \mathrm{A}f = \sqrt{2aa + \frac{9cc}{4}} + \frac{3c}{2}$. If $\mathrm{A}a = c = o$, then $\mathrm{A}t : \mathrm{AC} :: \sqrt{2} : 1$, or AP the sine of ACE to the radius AC as AG to 2CA, or as $\sqrt{\mathrm{AD}}$ to $\sqrt{2\mathrm{CA}}$, that is, as $\sqrt{2}$ to $\sqrt{3}$. Therefore the stream at the beginning of the motion will have the greatest effect upon the plane CE, if the angle ACE be of 54°. 44'.; and this is the case which has been considered by several authors: but if the plane CE has already a motion in the direction CB, the stream will have the greatest effect upon it if the angle ACE be greater. For example, if the velocity of the plane CE in the direction CB be a third part of the velocity of the stream, or $c = \frac{a}{3}$, then $\mathrm{A}t = \sqrt{2aa + \frac{aa}{4}} + \frac{a}{2} = 2a$, or the tangent of the angle ACE ought to be double of the radius, that is, ACE = 63°. 26'. If $c : a :: \sqrt{8} : \sqrt{9}$, then $\mathrm{A}t : \mathrm{AC} :: 2 + \sqrt{2} : 1$, and ACE ought to be of 73° 40'. If $c = a$, then ACE = 74° 19'.

914. Hence the sails of a common windmill ought to be so situated that the wind may strike them in a greater angle than

 that

that of 54° 44′; for this is the most advantageous angle at the beginning of the motion only; and when any part of the engine has acquired a velocity c, the effect of the wind upon that part will be greatest when the tangent of the angle in which the wind strikes it is to the radius as $\sqrt{2 + \frac{9cc}{4aa}} + \frac{3c}{2a}$ to 1. Let the right line bh represent the length of one of the sails, take AC to Ab as the velocity of the wind to the velocity of the given point b about the axis of motion, LA $=$ AC $\times \sqrt{2}$, and a being any point upon bh, take A$f = \frac{3\text{A}a}{2}$; then if the sail be so formed that the wind shall strike it at any distance Aa from the axis of motion in an angle whose tangent is always to the radius as Lf + Af to CA, the wind shall have the greatest effect upon the sail. It is true, that a celebrated author has drawn an opposite conclusion from his computations, viz. that the angle in which the wind strikes the sail ought to decrease as the distance from the axis of motion increases; that if $c = a$, the wind ought to strike the sail in an angle of 45°; and that if the sail be in one plane, it ought to be inclined to the wind at a *medium* in an angle of about 50 degrees: but if he had reduced the equation of six dimensions, by which he has determined the *maximum*, to a biquadratic equation, our conclusions would have agreed; and the divisor by which this reduction may be made is of no use for determining the most advantageous position of the sail when the engine is in motion; because it does not give a *maximum*, but a *minimum* that corresponds to the case when CE coincides with Ca, and the stream has no effect upon the plane CE. Suppose A$a =$ AC, or $c = a$; and if the angle ACE be of 45°, CE will coincide with Ca, the velocities of the plane CE and of the stream estimated in the direction perpendicular to CE must be equal; so that the stream will have no effect upon the plane CE in this case to preserve or accelerate its motion; and the angle ACE must be increased, that the velocity of the stream in the direction ap (in which it acts upon the plane) may be greater than the velocity of the plane in the same direction. In the same manner it is

is obvious that, if Aa was equal to 2AC, and ACE of 54° 44′. then the stream could have no effect upon the plane CE, and the angle ACE must be increased.

915. When (*fig.* 339) the engine is of such a nature that the whole fluid, or the same quantity of it, is always incident on the plane CE in its various positions, the force by which it impels CE in the direction CB is as $ak = Em \times EN \times \frac{Ca}{CE^2}$, which is a *maximum* (Ca and CE being given) when CE bisects the angle aCB, by art. 910, because in this case $n=1$, CD : CV :: $n-1 : n+1 :: o : 2$, that is, CD vanishes, and DG coincides with CB. In this case, if AC and Aa, the velocities of the stream and plane, be given, with CB the direction of the motion of the plane, but the angle ACB be variable, and Aa be greater than $\frac{1}{2}$ AC, the action of the fluid upon the plane will not be greatest when AC is perpendicular to CE and CE to CB; but when ACB being an obtuse angle, the sine of ACV is to the radius as AC to 2Aa, and the plane CE is perpendicular to AC. For let Cg = Ca, aq be perpendicular to CB in q, then $ak = \frac{1}{2}gq$. Suppose CA = a, Aa = c, AV = x, then ak = Cg ∓ Cq = Ca + aV = $\sqrt{aa+cc-2cx}+x-c$; and when the fluxion of this quantity vanishes, $\frac{-c\dot{x}}{\sqrt{aa+cc-2cx}}+\dot{x}=o$, $aa+cc-2cx=cc$, $aa=2cx$, or $x : a :: a : 2c$; and it is easy to see from the construction that in this case ACE must be a right angle. For example, if $c=a$ then $x=\frac{1}{2}a$, ACV = 30 degr. ACB = 120 degr. ACE = 90 degr. and ECB = 30. degr.

916. Suppose now that AC (*fig.* 338) represents the direction and velocity of the wind, CB the direction in which a ship moves, Aa parallel to CB the velocity of the ship, CE the situation of the sail, and let us abstract from her *leeward* way, or suppose that no deflexion from the direction CB is occasioned by the obliquity of the wind or sail to the course CB. Then, in order to determine the most advantageous position of the sail CE (when CA, CB, and Aa, are given), that the wind may act with the greatest force to impel the ship in the given direction CB, produce AC

till AD : AC : : 4 : 3. Let DG be parallel to CB, and a circle *aeg* described from the centre C with the distance Ca meet DG in g; then the sail CE ought to bisect the angle aCg, by Art. 911; or let CV be perpendicular to Aa in V, LV = VC × $\sqrt{2}$ Vf = $\frac{1}{2}$ Va, Vt = Lf + Vf, and CE produced pass through t. When Aa the velocity of the ship is neglected, or when the motion begins, CE ought to bisect the angle ACG; which is the case that was resolved long ago by Mr. *Fatio* and Mr. *Huygens* by a biquadratic equation; and has been considered more fully since by Mr. *Bernouilli, Manoeuvre des Vaisseaux*, chap. 5. But in some cases the ratio of Aa to AC is not inconsiderable; and supposing AC perpendicular to CB, if (for example) Aa = $\frac{1}{4}$ AC, the angle ACE ought to exceed $\frac{1}{2}$ ACG (= 54° 44′ in this case) by about $9\frac{2}{3}$ degr., if we would have the wind impel the ship with the greatest force in the direction CB.

917. The force with which the wind impels the ship in the direction CB is always measured by $ak \times ap$; and when this force and the resistance of the water become equal, the motion of the ship becomes uniform. Let CK represent the uniform velocity which the ship would acquire by the same wind in its direction AC, if the sail was perpendicular to AC, and the force in this case which sustains the motion of the ship, and balances the resistance, will be measured by KB². Therefore (the resistance of the water being as the square of the velocity of the ship) CK² : Aa^2 : : KB² : $ap \times ak$ = (supposing Aa parallel to CB to meet CE in t) $at^2 \times \frac{EN^3}{CE^3}$; consequently A$a$: at : : CK $\times \sqrt{\frac{EN^3}{CE^3}}$: KB. Let CA = a, Aa = x, EN = y, AV = p, CK : KB : : 1 : m; then Aa : at : : $y\sqrt{y}$: $ma\sqrt{a}$; and, because At = Vt ∓ AV = $\sqrt{aa-pp} \times \frac{\sqrt{aa-pp}}{y} \mp p$, A$a$ = x = $\frac{\sqrt{aa-pp} \times \sqrt{aay-aay} \mp py\sqrt{y}}{ma\sqrt{a} \mp y\sqrt{y}}$. Suppose a, p, and m, to be con-

stant,

stant, and when x is a *maximum* we shall find that $aa - 3yy - \frac{2y\sqrt{ay}}{m} \mp \frac{3py\sqrt{aa-yy}}{\sqrt{aa-pp}} = 0$. This is an equation for determining the sine of the angle ECB which ought to be contained by the sail and the line of the ship's motion, in order that the velocity of the ship in this line may be the greatest possible, a, p, and m, being given.

918. If AC be perpendicular to CB, then $p = 0$, and $3yy + \frac{2y\sqrt{ay}}{m} = aa$. For example, let $m = 2\sqrt{2}$, that is, let the velocity of the ship be to the velocity of the wind when the ship moves in the direction of the wind, and the wind is perpendicular to the sail as 1 to $1 + 2\sqrt{2}$ (or nearly as 1 to 3.828); then, if the ship sail in a direction perpendicular to that of the wind, the sail ought to be inclined to the wind in an angle of 60°, or to the way of the ship in an angle of 30°. For the equation for y when x is a *maximum* is, in this example, $aa - 3yy - \frac{y\sqrt{ay}}{\sqrt{2}} = 0$, which gives $y = \frac{a}{2}$; and in this case the velocity of the ship is $\frac{a\sqrt{y} \times \sqrt{aa-yy}}{ma\sqrt{a} + y\sqrt{y}} = \frac{a}{3\sqrt{3}}$. The sine of the angle ECB is always less than $\frac{1}{\sqrt{3}} \times$ CB.

919. The angle ECB (*fig.* 340) contained by the sail CE and course of the ship CB, with AC the velocity of the wind being given, the velocity of the ship is greatest when ACE is a right angle, that is, when the wind is perpendicular to the sail; as is obvious, and agrees with art. 917, where, if a, y, and m, be given, x becomes a *maximum* when $p = y$. Supposing AC to be perpendicular to CE, $x = \frac{aa\sqrt{y}}{ma\sqrt{a} + y\sqrt{y}}$, and is a *maximum* when y or $p = a \times \sqrt[3]{\frac{mm}{4}}$; that is, of all cases wherein the wind is supposed to be perpendicular to the sail, the velocity of the ship is greatest

 (providing

(providing CK be not less than $\frac{1}{2}$CA, or m be not greater than 2) when the sine of the angle ECB contained by the sail and course is to the radius as $\sqrt[3]{mm}$ to $\sqrt[3]{4}$, and the velocity of the ship is greater in this case than when the wind blows in the direction of the course, and is perpendicular to the sail in the ratio of $m + 1$ to $3\sqrt[3]{\frac{mm}{4}}$, or (supposing $n = 2 - m$) of $1 - \frac{n}{3}$ to $\overline{1 - \frac{n}{2}}|^{\frac{2}{3}}$. If, for example, CK : CA : : 1 : 2, the velocity of the ship in the direction CB will be greatest when the sine of ECB, or ACV, is to the radius as 1 to $\sqrt[3]{4}$; that is, when the angle ECB is about 39° 3′, or when the angle AC*b*, in which the direction of the wind is inclined to the course of the ship, is an angle of about 50° 57′. And the velocity of the ship is in this case greater than when the same wind blows directly in the course of the ship, and the sail is perpendicular to the wind (in which case the wind is commonly thought to be most favourable) in the ratio of $\sqrt[3]{32}$ to $\sqrt[3]{27}$, or of $2\sqrt[3]{4}$ to 3; and by inclining the sail CE to the wind, so as to increase the angle BCE, the velocity of the ship in the right line CB will be still greater. There may be many other cases supposed from art. 916, wherein a side-wind would promote the motion of the vessel more than a direct wind. For example, if the velocity of the vessel in the direction CB be to the velocity of the wind as 1 to 3, and the angle ACB be only of 109° 28′, the force by which the wind will promote the motion of the vessel in the course CB will in this case be greater than when the wind is direct, or the angle ACB is of 180°, in the ratio of $\sqrt[3]{32}$ to $\sqrt[3]{27}$; the sail being supposed in both cases to have the most advantageous position, which was determined in art. 916.

920. A given line AC (*fig.* 341) being divided in B, the rectangle AB × BC is a *maximum* when AB = BC, by what is shown in the elements of geometry. Hence it follows, that, if a given line

AG

AG be divided into a given number of parts AB, BC, CD, DE, EF, FG, the product of the parts AB × BC × CD × DE, &c. is a *maximum* when they are equal to each other; because if BD the sum of any two adjoining parts be divided equally in C and inequally in *c*, BC × CD is greater than Bc × cD, and AB × BC × CD × DE × &c. is greater than AB × Bc × cD × DE × &c. If a given right line AG be divided in C, and $AC^n \times CG^m$ be a *maximum*, then AC : CG : : *n* : *m*; for if we suppose AG $= a$, AC $= x$, $x^n \times \overline{a-x}^m = y$, then $\frac{\dot{y}}{y} = \frac{n\dot{x}}{x} - \frac{m\dot{x}}{a-x}$, and if $\dot{y} = o$, $\frac{n}{x} = \frac{m}{a-x}$, that is, AC : CG : : *n* : *m*. The same proposition may be derived from the former case when *n* and *m* are any integer numbers: for example, AB × BG^5 is a *maximum* when AB is to BG as 1 to 5; because if BG be divided into five equal parts BC, CD, &c. then AB × BG^5 $= 5 \times 5 \times 5 \times 5 \times 5 \times$ AB × BC × CD × DE × EF × FG, which is a *maximum* when AB = BC = CD = DE = EF = FG. If AG be divided into three parts AB, BD, and DG, then AB × $BD^n \times DG^m$ is a *maximum* when AB, BD, and DG, are to each other in the same proportion as the numbers 1, *n*, and *m*, respectively; because, wherever we suppose the point B to be, $BD^n \times DG^m$ cannot be a *maximum* (and consequently AB × $BD^n \times DG^m$ is not a *maximum*) unless BD : DG : : *n* : *m*; and wherever we suppose the point D to be, AB × BD^n cannot be a *maximum* unless AB : BD : : 1 : *n*. The continuation of those theorems is obvious; and this brief method of resolving several questions relating to *maxima* and *minima* that cannot be so easily reduced to the common rules, was mentioned in a letter to *Martin Folkes, Esq. Phil. Trans.* No. 408. The following article gives another useful instance.

921. The radius AC (*fig.* 342) and ark AF being given, let AF be divided into three parts, AE, EB, and BF, let EM, EN, and BR, be the sines of the arks AE, EB, and BF; then if $EM^n \times EN \times BR^m$ be a *maximum*, the tangents of the arks AE, EB, and BF,

BF, shall be in the same proportion as the numbers n, 1, and m. This follows from art. 910, because, wherever we suppose the point B to be placed upon the ark FE, $EM^n \times EN$ is not a *maximum* (art. 910), unless the tangent of the ark AE be to the tangent of EB as n to 1; consequently the ark AB must be divided in this manner, that $BR^m \times EN \times EM^n$ may be a *maximum*. In like manner, wherever we suppose the point E to be taken upon the ark AB, $EN \times BR^m$ cannot be a *maximum*, unless the tangent of EB be to the tangent of BF as 1 to m; and the ark FE must be divided in this manner, that $EM^n \times EN \times BR^m$ may be a *maximum*. Therefore if $EM^n \times EN \times BR^m$ be a *maximum*, the tangent of AE must be to the tangent of EB as n to 1, and the tangent of EB to the tangent of BF as 1 to m; that is, the ark FA must be divided in such a manner in B and E that the tangents of AE, EB, and BF, may be in the same proportion to each other as the numbers n, 1, and m. If $n = m$, then AE = BF. The continuation of these theorems is likewise obvious. If a given ark be divided into any given number of parts whose sines are represented by a, b, c, d, e, &c. and $a^m \times b^n \times c^r \times d^s \times$ &c. be a *maximum*, then the tangents of the respective parts must be in the same proportion as the indices, m, n, r, s, &c. and (because the sine of an ark is to the radius as the radius to the secant of the same ark) the product of the same powers of the respective secants of those parts is a *maximum*.

928. For an example of this, the force and direction of the wind being given, let it be required to find the most advantageous course of the ship and position of the sail, that the ship may be carried in a given direction, or removed from a given coast or right line, as fast as possible. Let AC represent the force and direction of the wind, CF the line from which the ship is to be carried as fast as possible, CB the course of the ship, and CE the position of the sail. Let AQ be parallel to CB, AP perpendicular to CE in P, and PQ perpendicular to AQ in Q. Then the force by which the wind impels the ship in the direction CB at the beginning of the motion will be as $AP \times AQ$ =

$= EM^2 + \frac{EN}{CE}$; and the velocity of the ship (supposing it to be incomparably less than the velocity of the wind) shall be as $EM \times \sqrt{EN}$; which, reduced to the direction BR perpendicular to CF, is as $EM \times \sqrt{EN} \times BR$; and this last velocity is a *maximum* (by the last article) when the tangents of the arks AE, EB, and BF, are in the same proportion as the numbers 1, $\frac{1}{2}$ and 1, or 2, 1 and 2. Let the radius $CE = a$, the tangent of AF be represented by b, the tangent of AE or BF by t, and the tangent of AB or FE by T. Because the arks $AB + BF = AF$, it will easily appear that $t = a \times \frac{ab - aT}{aa + bT}$; and in the same manner, because $BF + BE = FE$, the tangent of BE $(= \frac{t}{2}) = a \times \frac{aT - at}{aa + Tt}$; whence $T = \frac{3aat}{2aa - tt}$; consequently $t^3 - 4btt - 5aat + 2baa = o$; and, b being given, t and T may be found by the resolution of this cubic equation.

923. If FCA be a right angle, then b is infinite, and $2tt = aa$, or $t : a :: 1 : \sqrt{2}$, and $T : a :: \sqrt{2} : 1$; that is, $ACB = FCE = 54° 44'$; consequently, if the velocity of the ship may be neglected as incomparably less than the velocity of the wind, the course ought to contain an angle of 54° 44′, and the sail an angle of 35° 16′ with the direction of the wind, that the ship may gain upon the wind as much as possible; and this is the case resolved by Mr. *Bernouilli, Manoeuvre des Vaisseaux,* p. 50, &c. If the course CB and position of the sail CE is required, that the ship may get away from the line AC as fast as possible, then we are to suppose ACF to be a continued right line, or $b = o$, in which case $tt = 5aa$, or $t : a :: \sqrt{5} : 1$; consequently the angle ACE ought to be of 65° 54′, and ACB of 114° 6′. If the angle ACE be given, the tangent of ECB ought to be to the tangent of ECF as 2 to 1; and ECB is determined by a construction similar to that in art. 910. We have always supposed the sail to be a plane, and have abstracted from the lee-way of the ship, but shall not enter farther into this

this theory at present. Mr. *Renau* published an ingenious treatise on this subject in 1689; but some particulars in it have been corrected by Mr. *Huygens* and Mr. *Bernouilli*. Several other mechanical problems may be resolved in the same manner as these we have considered.

924. In book I. chap. 13, it was shown how many problems may be immediately reduced to equations that involve first fluxions only, which it has been usual to resolve first by equations that involve second or higher fluxions; but as that method is not always applicable, we shall give some examples of the method of reducing equations from second to first fluxions. Suppose $\dot{x}$ constant, and if the equation involve $\dot{x}$, $\dot{y}$, and $\ddot{y}$, but if either x or y be wanting (of which kind are those which arise most commonly in the resolution of problems), it may be reduced to first fluxions, by introducing a new variable quantity z, and supposing it equal to $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$. Suppose, for example, that $\dot{x}^2 + \dot{y}^2 = \frac{y\ddot{y}}{n}$, let $\dot{y} = z\dot{x}$, and consequently $\ddot{y} = \dot{z}\dot{x}$, then $n\dot{x}^2 \times \overline{1+zz} = y\dot{z}\dot{x}$, or $n\dot{x} \times \overline{1+zz}$ $(= n\dot{y} \times \frac{1+zz}{z}) = y\dot{z}$; therefore $\frac{2n\dot{y}}{y} = \frac{2z\dot{z}}{1+zz}$, and (by art. 740) $y^{2n} = \overline{1+zz} \times A$, or $zz = \frac{y^{2n}}{A} - 1 = \frac{\dot{y}^2}{\dot{x}^2}$; consequently $\dot{x} = \frac{\dot{y}\sqrt{A}}{\sqrt{y^{2n}-A}}$, where A denotes an invariable quantity.

925. Let the point T (*fig.* 343) move in the right line A*a*, and the point M in the curve FM, so that the velocity of the point T may be to the velocity of the point M in the invariable ratio of n to 1, and the motion of M may be always in the direction MT or TM; and let it be required to determine the curve FM. Let AP $= x$, PM $= y$, FM $= s$, AT $= t$; then $n\dot{s} = \dot{t} =$ (because $t = x - \frac{y\dot{x}}{\dot{y}}$, or to $\frac{y\dot{x}}{\dot{y}} - x$, and $\dot{x}$ is supposed constant)

$\mp\frac{\ddot{y}y\dot{x}}{\dot{y}^2}$

$\frac{\mp y\ddot{y}\dot{x}}{\dot{y}^2}$. Let $\dot{x} = z\dot{y}$, then $\dot{z}\dot{y} + z\ddot{y} = 0$, and $n\dot{y}\sqrt{1+zz} = \frac{\pm y\dot{y}\dot{z}\dot{x}}{z\dot{y}^2} = \pm y\dot{z}$, and $\frac{n\dot{y}}{y} = \frac{\pm\dot{z}}{\sqrt{1+zz}}$; whence $Ay^n = \sqrt{1+zz} \pm z$, and $AAy^{2n} \mp 2Azy^n = 1$, or $2z = \frac{2\dot{x}}{\dot{y}} = \frac{\mp 1}{Ay^n} \pm Ay^n$, consequently $2\dot{x} = \frac{\mp\dot{y}}{Ay^n} \pm Ay^n\dot{y}$, and $2x = \frac{\mp 1}{\overline{1-n} \times Ay^{n-1}} \frac{\pm Ay^{n+1}}{n+1} + K$, where K denotes an invariable quantity. If $n = \frac{1}{2}$, then $x = \frac{\mp\sqrt{y}}{A} \pm \frac{Ay\sqrt{y}}{3} + K$, and the curve is a parabola of the third order of lines. If $n = 1$, the curve (*fig.* 344) is constructed by logarithms or the equilateral hyperbola. Let NDN be such an hyperbola described betwixt the asymptotes C*a* and C*b*, D a given point in the hyperbola, join CD, let NLM perpendicular to the asymptote C*a* in L meet CD in K, and let LM × 2FD be always equal to the area DNK; then M shall be a point in the curve.

926. An equation that involves second fluxions is sometimes easily reduced to first fluxions, by the common rules of the inverse method, which were described in chap. 2; and that the solution may be general enough, when any fluxion is supposed constant, a quantity compounded from it or from its powers and invariable quantities ought to be added to the equation. For example, let it be required to find the nature of the line in which the curvature is always as the ordinate, this being a figure by which several problems of different kinds are resolved. Let the ray of curvature be represented by R, and because $R = \frac{\dot{y}\dot{s}^2}{\dot{s}\ddot{x} - \ddot{x}\dot{s}}$, suppose $\dot{s}$ constant, then $R = \frac{\dot{y}\dot{s}}{\ddot{x}}$. In the figure required R is inversely as the ordinate y; consequently, a being an invariable quantity, we may suppose $\frac{aa}{2y} = R = \frac{\dot{y}\dot{s}}{\ddot{x}}$

or

or $2\dot{y}y\dot{s} = aa\ddot{x}$; and by finding the fluents, $yy\dot{s} = aa\dot{x} + K\dot{s}$ where K denotes an invariable quantity, and $K\dot{s}$ is added because $\dot{s}$ is supposed constant. If $K = o$, then $\dot{s}^2 : \dot{x}^2 :: a^4 : y^4$, $\dot{y}^2 : \dot{x}^2 :: a^4 - y^4 : y^4$, and consequently $\dot{x} = \frac{\mp yy\dot{y}}{\sqrt{a^4 - y^4}}$.

927. The celebrated author who first resolved this as well as several other curious problems, after his account of this figure (which is commonly called the *elastic curve*), adds, *Ob graves causas suspicor curvæ nostræ constructionem a nullius sectionis conicæ seu quadratura seu rectificatione pendere, Act. Lips.* 1694, p. 272. But it is constructed by the rectification of the equilateral hyperbola; for if the base of a figure be always taken equal to the perpendicular from the centre on the tangent of such an hyperbola, and the ordinate equal to the excess of the tangent terminated by that perpendicular above the ark intercepted betwixt the vertex of the hyperbola and the point of contact, then the figure shall be the elastic curve. Let AEZ (*fig.* 345) be an equilateral hyperbola that has its centre in S and vertex in A, let E be any point in the hyperbola, ET a tangent at E, and SP perpendicular from the centre S to the tangent at P; upon SA take SQ = SP, and the ordinate QM always equal to the excess of the tangent EP above the ark AE of the hyperbola; then M shall be a point in the elastic curve AMB. For suppose SA = a, SQ (= SP) = y, QM = x, SE = r, EP = z, and the ark AE = s, then $r = \frac{aa}{y}$, EP = $z = \sqrt{rr - yy} = \frac{\sqrt{a^4 - y^4}}{y}$, $\dot{z} = \frac{-y^4\dot{y} - a^4\dot{y}}{yy\sqrt{a^4 - y^4}}$. But $\dot{s} : \dot{r} :: r : z :: aa : \sqrt{a^4 - y^4}$ and $\dot{r} = \frac{-aa\dot{y}}{yy}$; consequently $\dot{s} = \frac{-a^4\dot{y}}{yy\sqrt{a^4 - y^4}}$, and $\dot{x} = \dot{z} - \dot{s} = \frac{-y^4\dot{y} - a^4\dot{y} + a^4\dot{y}}{yy\sqrt{a^4 - y^4}} = \frac{-y^2\dot{y}}{\sqrt{a^4 - y^4}}$, which is the equation for the common elastic curve.

928. In

928. In general, the equation for the elastic curve was $aa\dot{x} = yy\dot{s} - K\dot{s}$; consequently $\dot{x} = \mp \dot{y} \times \frac{yy - K}{\sqrt{\overline{aa - K + yy} \times \overline{aa + K - yy}}}$; and by comparing this fluxion with those described in art. 804 and 805, it will appear that the elastic curve may be constructed in all cases by the rectification of the conic sections. Let SA (*fig.* 346) be half the transverse axis of the hyperbola AEH, SD half the second axis; upon DS take SF. SA : : SA. SD, and Sb $=$ AF, describe the elliptic quadrant ARb, and, E being any point in the hyperbola, SP perpendicular to the tangent EP in P, upon SA take SQ $=$ SP, and let the ordinate QR meet the ellipse in R; then, by taking QM upon QR equal to $\frac{AD^2}{2SD^2} \times \overline{EP - AE} + \frac{SD^2 - SA^2}{2SA \times SD} \times AR$, M shall be a point in the elastic curve; and the ray of curvature at any point M shall be equal to $\frac{AD^2}{4SQ}$, because in comparing those fluxions we suppose $aa - K = SD^2$ and $aa + K = SA^2$, or $2aa = SD^2 + SA^2 = AD^2$, and the ray of curvature was supposed equal to $\frac{aa}{2y} = \frac{AD^2}{4SQ}$.

929. Let SA (*fig.* 347) be incomparably less than SD, then, because SD : SA : : SA : SF, we may suppose SF to vanish, ARb to be a quadrant of a circle, and EP — AE to vanish; consequently QM $= \frac{SD}{2SA} \times$ AR and SB $= \frac{SD}{2SA} \times$ ARb $=$ (if the ratio of m to 1 denote that of the circumference to the diameter) $\frac{m}{4} \times$ SD; and the elastic curve in this case will represent the figure which a musical chord BAC assumes in its small vibrations by the converse of art. 569, the tension of the chord being every where equal. Let P represent this tension, or a weight equivalent to it, n a section of the chord perpendicular to its length, R the ray of curvature at any point M, and V the force by which the motion of any point at M towards BC is accelerat-

ed,

ed, while the chord returns to its natural state; then, by art. 561, the tension would be equal to the weight of a chord of the same thickness of the length R, if the gravity was equal to V; that is, $P = nRV$, or $V = \frac{P}{nR} = \frac{P}{n} \times \frac{4SQ}{AD^2} =$ (because SD is to AD nearly in a ratio of equality) $\frac{P}{n} \times \frac{4SQ}{SD^2} = \frac{mmP}{n} \times \frac{SQ}{4SB^2}$. If we suppose, with Dr. *Taylor*, N to represent the weight of the chord, L its length, g the force of gravity, D the length of a given pendulum, $SA = a$, SQ or $MN = y$; then, because $N = nLg$, or $n = \frac{N}{Lg}$, and $L = 2SB$, it follows, that $V = \frac{mmPgy}{NL}$. Because V is as y the distance of M from BC, the vibrations of the chord are similar to those of a pendulum; and the time in which M describes MN is to the time in which the pendulum D performs half a vibration as $\sqrt{\frac{y}{V}}$ to $\sqrt{\frac{D}{g}}$, or as $\frac{\sqrt{LN}}{m\sqrt{P}}$ to $\sqrt{D}$; consequently the number of vibrations made by the chord, while the pendulum vibrates once, is expressed by $m \times \sqrt{\frac{DP}{LN}}$, which is *Dr Taylor*'s theorem, and serves for determining the number of vibrations made in a given time by any given chord that is extended by a given weight; or for comparing the number of vibrations made by different chords in equal times, upon which their tone depends. Thus if the weight P be the same, the number of vibrations is as $\frac{1}{\sqrt{LN}}$; and when the chords are of the same kind (or N is as L) the vibrations are as $\frac{1}{L}$. If the chord be given, the number of vibrations is as $\sqrt{P}$. The ratio of m to 1, or of the circumference to the diameter, enters the expression of the number of the vibrations of the chord; because the ratio of 2SB the length of the chord to the ray of curvature involves it; and there seems to be a difference

ference in this respect betwixt the theorems by which the vibrations of a musical chord, and these which are produced in the air by organ-pipes, or other wind-instruments, are to be determined.

930. Because the elastic curve is defined by the equation (art 926), $y\dot{y}\dot{s} - K\dot{s} = aa\dot{x}$, it follows, that $\dot{s} = \frac{\mp aa\dot{y}}{\sqrt{a^4 - \overline{yy - K}^2}}$ $= \frac{\mp aa\dot{y}}{\sqrt{aa + K - yy} \times \sqrt{aa - K + yy}}$. Let this fluxion be compared with that in art. 805 (*fig.* 346), of which we found the fluent to be $\frac{SD}{SA} \times AR + AE - EP$; and it will appear, by supposing $aa + K = SA^2$, and $aa - K = SD^2$, that AM the ark of the elastic curve is equal to $\frac{AD^2}{2SD^2} \times \overline{\frac{SD}{SA} \times AR + AE - EP}$. Therefore the figures that have been constructed by the rectification of the elastic curve may be constructed by the rectification of the hyperbola and ellipsis; particularly the curve along which if a heavy body moved it would recede equally in equal times from a given point, which Mr. *Leibnitz* constructed by the rectification of a geometrical curve of a higher kind than the conic sections, and Mr. *James Bernouilli* by the elastic curve, *Act. Lips.* 1694, p. 272, 277, 338, 370, &c. The fluents of $\frac{z^2\dot{z}}{\sqrt{z^4 - a^4}}$, $\frac{a^2\dot{z}}{\sqrt{z^4 - a^4}}$, $\frac{a^2\dot{z}}{\sqrt{a^4 - z^4}}$, and $\frac{z^2\dot{z}}{\sqrt{a^4 - z^4}}$ (which are mentioned, *ibid*, p. 338, where it is said of the first only, that it may be assigned by the rectification of the hyperbola) are all assignable by the rectification of the equilateral hyperbola, and of the ellipsis, whose excentricity is equal to the second axis. Let AE and AR (*fig.* 348) be such an hyperbola and ellipsis, $SA = a$, and $SE = z$, then the F. $\frac{z^2\dot{z}}{\sqrt{z^4 - a^4}} = AE$, and the fluent of $\frac{aa\dot{z}}{\sqrt{z^4 - a^4}} = AR + AE - EP$. If SP be perpendicular from the

the centre S on the tangent at E in P, SA $= a$, and SP $= z$, then the F. $\frac{-a^2\dot{z}}{\sqrt{a^4-z^4}} =$ AR + AE — EP, and the F. $\frac{-z^2\dot{z}}{\sqrt{a^4-z^4}}$ $=$ EP — AE, as appears from art. 799 and 802. Fluents of other forms may be assigned by the rectification of the conic sections by art. 804 and 805.

931. It may be worth while to show here how the same easy method which was described in chap. 13, book I. for determining, by first fluxions only, the nature of the lines of swiftest descent, of the figures that amongst all those of equal perimeters produce *maxima* and *minima*, and of that which generates the solid of least resistance, serves with equal facility and evidence for discovering the equation of the curve when other limitations are added in the problem; as when it is required to find the solid, which amongst all those of equal capacities, and that are bounded by equal surfaces, meets with the least resistance in a fluid. The fundamental *lemma* (demonstrated in art. 572 and 592) is that, if AK (*fig.* 349) be given, KE be perpendicular to AK, a and u denote any given or invariable quantities, then AE $\times a$ — KE $\times u$ (or $\frac{AE}{u} - \frac{KE}{a}$) is a *minimum* when KE : AE : : $u : a$, or $a \times$ KE $= u \times$ AE. Let the base FP $= x$, the ordinate PA $= y$, the ark GA $= s$, AK $= \dot{y}$, and if AE the tangent at A meet KE parallel to the base in E, then AE $= \dot{s}$ and KE $= \dot{x}$; and it follows from the *lemma*, that if V and u represent any quantities compounded from the powers of y (so as to be of the same value when y is the same), and if $\dot{y}$ be given, then $V\dot{s} - u\dot{x}$, and $\frac{\dot{s}}{u} - \frac{\dot{x}}{V}$ are *minima* when $V\dot{x} = u\dot{s}$. From this it follows (as in art. 576 and 593), that if GAD be the whole curve, and DH the difference of the ordinates at G and D be given, then the F. $V\dot{s}$—F. $u\dot{x}$, or the F. $\frac{\dot{s}}{u}$ — F. $\frac{\dot{x}}{V}$, shall be a *minimum* when the nature of the figure is defined by the equation $V\dot{x} = u\dot{s}$. Therefore, supposing this

this to be the equation of the curve, and DH to be given, if the fluent of $V\dot{s}$ be also given, then the F. $u\dot{x}$ shall be a *maximum*; or if the latter fluent be given, then the former shall be a *minimum*: and if the fluent of $\frac{\dot{s}}{u}$ be given, the F. $\frac{\dot{x}}{V}$ shall be a *maximum*; or if the fluent of $\frac{\dot{x}}{V}$ be given, the fluent of $\frac{\dot{s}}{u}$ shall be a *minimum*. It appears, likewise (as in art. 595), that if DH with the base FC or GH be given, and the fluent of $V\dot{s}$ be given or invariable, then the F. $u\dot{x}$ will be a *maximum* or *minimum* when the equation of the curve is $V\dot{x} = \overline{e \mp u} \times \dot{s}$, where e denotes an invariable quantity that may be positive, or negative, or vanish.

932. Suppose, therefore, $V = A + By + Cyy + Dy^3 + \&c.$ and $u = a + by + cy^2 + dy^3 + \&c.$ where A, B, C, &c. and a, b, c, &c. denote any invariable coefficients that may be positive or negative, any of which may be supposed to vanish; and the fluent of $V\dot{s} - u\dot{x}$, that is, of $\dot{s} \times \overline{A + By + Cyy + \&c.} - \dot{x} \times \overline{a + by + cyy + \&c.}$ shall be a *minimum* when the equation of the figure is $\dot{x} \times \overline{A + By + Cyy + \&c.} = \dot{s} \times \overline{a + by + cyy + \&c.}$ the ordinate DH being given. Therefore, if the fluent of $\dot{s} \times \overline{A + By + Cyy + \&c.}$ be also given, the fluent of $\dot{x} \times \overline{a + by + cyy + \&c.}$ shall be a *maximum*; or if the latter be given, the former shall be a *minimum*: and if the base FC or GH be given with DH and the F. $\dot{s} \times \overline{A + By + Cyy + \&c.}$ then the F. $\dot{x} \times \overline{a + by + cyy + \&c.}$ shall be a *maximum* or *minimum* when $\dot{x} \times \overline{A + By + Cyy + \&c.} = \dot{s} \times \overline{e \mp A + By + Cyy + \&c.}$ Of which theorem it is an obvious but a particular case only, that, if the nature of the figure be defined by the last equation, and GH, DH, with the fluents of $A\dot{s}$, $By\dot{s}$, $Cyy\dot{s}$, &c. $by\dot{x}$, $cyy\dot{x}$, $dy^3\dot{x}$, &c. be all given or invariable, one only excepted, this last fluent shall be either a *maximum* or *minimum*.

933. For example, the points G and D being given, if the perimeter GAD (or the F. $A\dot{s}$) be also given, the area FGADC (or the F.$y\dot{x}$) is a *maximum* or *minimum* when $A\dot{x} = \dot{s} \times \overline{e+A+By}$, that is, when GAD is an ark of a circle. If the surface generated by the ark GAD about the axis FC (or the F. $By\dot{s}$) be given, then the solid generated by the figure FGADC about the same axis (or the F. $cyy\dot{x}$) is a *maximum* or *minimum* when $By\dot{x} = \dot{s} \times \overline{e+cyy}$; and when $e = o$, this is again a circle. If the perimeter GAD (or the F. $A\dot{s}$) be given, the solid generated by FGADC about the axis FC is a *maximum* or *minimum* when $A\dot{x} = \overline{e+cyy} \times \dot{s}$, and GAD is the elastic curve, which was constructed by the arks of conic sections in art. 928. If the F. $\dot{s} \times \overline{A+By}$ be given, then the fluent of $\dot{x} \times \overline{a+by+cyy}$ is a *maximum* or *minimum* when $\dot{x} \times \overline{A+By} = \dot{s} \times \overline{e+a+by+cyy}$. And it is no more but a particular case of this theorem that the same equation comprehends that of the figure when the points G, D, with the surface generated by GAD about FC and the area FGADC, are given or invariable, and the solid generated by this area about the axis FC is a *maximum* or *minimum*. For since, by the supposition, the fluents of $\dot{s} \times \overline{A+By}$ with the F. $a\dot{x}$ and F.$by\dot{x}$ are given, so that the F. $cyy\dot{x}$ alone is variable, and the fluent of F. $\dot{x} \times \overline{a+by+cyy}$ is a *maximum* or *minimum*, it is manifest that the F. $cyy\dot{x}$ is a *maximum* or *minimum*. Nor is there any occasion, in order to obtain such equations, to have recourse to higher fluxions, or to resolve the element of the curve into a number of infinitesimal parts. Other examples may be derived in the same manner at pleasure.

934. The same method is extended to several other sorts of problems, by art. 605. Let V be now compounded from the powers of $\dot{s}$ and $\dot{y}$ as well as from the powers of y and invariable quantities. For example, let $V = \frac{K\dot{y}^n}{\dot{s}^n} + A + By + Cy^2 + Dy^3 + \&c.$ where K is supposed to be compounded from the powers of y and invariable quantities, and $u = a + by + cyy$

cyy + &c. as formerly. Then it appears, as in article 605, that $V\dot{s} \mp u\dot{x}$ shall be a *minimum* when $\overline{1-n} \times \frac{K\dot{y}^n\dot{x}}{\dot{s}^n} + A\dot{x} + By\dot{x} + Cy^2\dot{x} + \&c. = \mp \dot{s}u = \mp \dot{s} \times \overline{a + by + cyy} + \&c.$ and by substituting 3 for n this equation serves for resolving the problems that may be proposed concerning the solid of least resistance. For, supposing the solid that is generated by the revolution of the figure FGADC (*fig.* 350) to move in a fluid with a given velocity, and in the direction of the axis CF, then, according to the common doctrine of the action of the particles of fluids on bodies (or if the fluid be rare, as Sir *Isaac Newton* supposes), the resistance of the conical surface generated by the tangent AE will be ultimately as $PA \times AK \times \frac{AK^2}{AE^2} = \frac{y\dot{y}^3}{\dot{s}^2} = \frac{y\dot{y}^3}{\dot{s}^3} \times \dot{s}$, and $2y\dot{y}^3\dot{x} - \dot{x}\dot{s} \times \overline{A + By + Cyy} + \&c. = \dot{s}^4 \times \overline{a + by + cyy} + \&c.$ is the equation for the curve that generates the solid of least resistance, when the points G and D with the fluents of $A\dot{s}$, $By\dot{s}$, $Cyy\dot{s}$, and the F. $\dot{x} \times \overline{a + by + cyy} + \&c.$ are supposed to be given or invariable. Thus if the points G and D only are given, the equation is $2y\dot{y}^3\dot{x} = a\dot{s}^4$, as Sir *Isaac Newton* found. If the solid of the least resistance is required amongst all the solids of equal capacity, the equation is $2y\dot{y}^3\dot{x} = a\dot{s}^4 + cy^2\dot{s}^4$. If the solids are supposed to be bounded by equal surfaces, the equation for the figure which generates the solid of least resistance is $2y\dot{y}^3\dot{x} - By\dot{x}\dot{s}^3 = a\dot{s}^4$. If the solid is to have the least resistance of all those that have equal capacity, and are terminated by equal surfaces, the equation is $2y\dot{y}^3\dot{x} - By\dot{x}\dot{s}^3 = a\dot{s}^4 + cy^2\dot{s}^4$; and in like manner the equation is found, when other limitations that relate to the perimeter GAD, area FGADC, &c. are superadded.

935. Because the theorems proposed in art. 563, and explained in the subsequent articles, are of more general use, it may be proper to give one example of the manner of applying them for

for discovering the equation of the figure required. Let u the velocity acquired at any point A be as $\overline{Ax^n + By^m} \times x^k y^l$, and the equation of the line of swiftest descent be required. Let OA (*fig.* 351) the ray of curvature at A be considered as given in position, and, supposing the point O to remain, let A move in the right line OA, and AP be always perpendicular to FC in P; let OA $= q$, FP $= x$, and AP $= y$, then if OA meet FC in I — $\dot{x} : \dot{q}$ :: PI : IA :: $\dot{y} : \dot{s}$, and $\dot{y} : \dot{q}$ :: PA : IA :: $\dot{x} : \dot{s}$. But, by the theorem in art. 565, OA and u increase proportionally while the point A is supposed to move in the right line OA, that is, $\frac{\dot{q}}{q} = \frac{\dot{u}}{u} = \frac{nAx^{n-1}\dot{x} + mBy^{m-1}\dot{y}}{Ax^n + By^m} + \frac{k\dot{x}}{x} + \frac{l\dot{y}}{y}$.

Hence by substituting $\frac{-\dot{q}\dot{y}}{\dot{s}}$ for $\dot{x}$, and $\frac{\dot{q}\dot{x}}{\dot{s}}$ for $\dot{y}$, then dividing by $\dot{q}$, and substituting for the ray of curvature q its value $\frac{\dot{s}^3}{-\dot{x}\ddot{y}}$, $\dot{x}$ being supposed constant, it follows, that $\frac{\ddot{y}\dot{x}}{\dot{s}^2} = \frac{nAx^{n-1}\dot{y} - mBy^{m-1}\dot{x}}{Ax^n + By^m} + \frac{k\dot{y}}{x} + \frac{l\dot{x}}{y}$. If it be required that the curve shall be described in less time than any other of an equal perimeter, the equation may be found by the third general principle described in art. 563.

936. The preceding examples may serve to show the extensive usefulness of the method of fluxions in geometry and the various parts of philosophy. In the account we gave of this doctrine in the first book, we supposed with Sir *Isaac Newton* quantities to be generated by motion, and considered the fluxion of a quantity as the velocity or measure of this motion. Some propositions, however, were demonstrated (as prop. 20 and 32) without making use of fluxions; and several other theories described in this and the preceding book may be likewise established in a manner independent of the notion of a flux-

ion.

ion. Thus, it is easily demonstrated from art. 710, that, supposing n to be any integer and positive number, if the area upon the base AP (*fig*. 352) or x be always equal to x^n, then the ordinate PM or y shall be always equal to nx^{n-1}. For let o represent Pp any increment of the base x; and, because x and y increase together, PM $\times$ Pp, or $y \times o$, shall be less than PMmp $= \overline{x+o}^n - x^n$, the simultaneous increment of the area, which (by substituting $x + o$ for E, and x for F, in art. 710) is less than $n \times \overline{x+o}^{n-1} \times o$; consequently y is less than $n \times \overline{x+o}^{n-1}$. In the same manner, it appears that PM $\times$ Pp, or $y \times o$, is greater than PM$\mu\pi$ $= x^n - \overline{x-o}^n$, which, by the same article, is greater than $no \times \overline{x-o}^{n-1}$; consequently the ordinate y is greater than $n \times \overline{x-o}^{n-1}$. But if y be said to be greater than nx^{n-1}, suppose $y = nx^{n-1} + r$, and $o = \overline{x^{n-1} + \frac{r}{n}}\Big|^{\frac{1}{n-1}} -$ x, or $\overline{x+o}^{n-1} = x^{n-1} + \frac{r}{n}$, then $y = nx^{n-1} + r = n \times$ $\overline{x+o}^{n-1}$, the contrary of which has been demonstrated; and if y be said to be less than nx^{n-1}, suppose $y = nx^{n-1}$ $- r$, and $o = x - \overline{x^{n-1} - \frac{r}{n}}\Big|^{\frac{1}{n-1}}$, or $\overline{x-o}^{n-1} = x^{n-1} -$ $\frac{r}{n}$, then $y = n \times \overline{x-o}\big|^{n-1}$, against what has been demonstrated: therefore $y = nx^{n-1}$. I intended to have subjoined demonstrations of this kind of some other theorems; but this seems to be unnecessary, after what has been shown at so great length in the first book, and the first chapter of this book, for demonstrating the evidence of this method. Sometimes we have spoke of infinites in this chapter in the usual style of writers on this subject; but we took no greater liberty in making use of such expressions than is allowed to authors in the inferior parts of these sciences, particularly to such

such as treat of trigonometry, who, while they assign a tangent and secant to every ark, and find that no finite tangent or secant can belong to the quadrant, therefore mark it *infinite* in their tables. In the same sense $\dot{y}$ or $\ddot{y}$ are in certain cases supposed to become infinite; but we pretend to draw no conclusions concerning infinites from the use of such concise and convenient expressions, nor inferences of any kind, but such as may be otherwise justified by unexceptionable evidence.

937. In this doctrine, when the velocity of a motion is determined, it is always with relation to the velocity of some other motion; and when we enquire at what rate the ordinate, for example, increases or decreases, it is always in relation to the base, or some other magnitude, with which it is compared. It is only relative space and motion we have occasion to consider in this method, than which no sort of quantities seem to be more clearly conceived by us.

FINIS.

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Plate XXXVII To be plac'd at the End of Vol II.

Fig. 320.

Fig. 318.

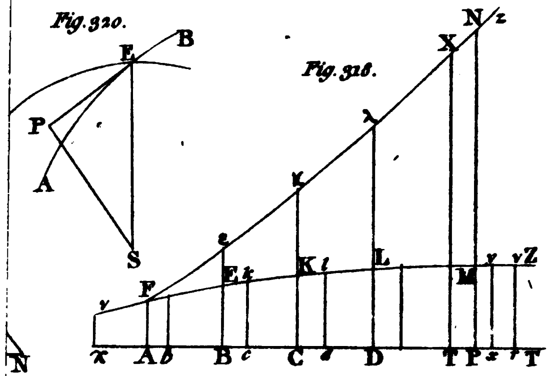


Fig. 322. N. 2.

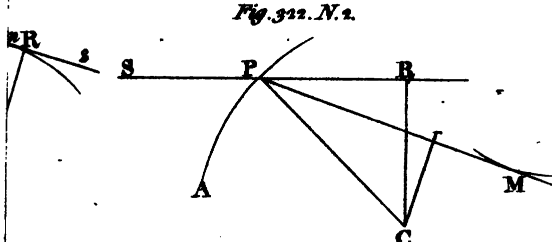
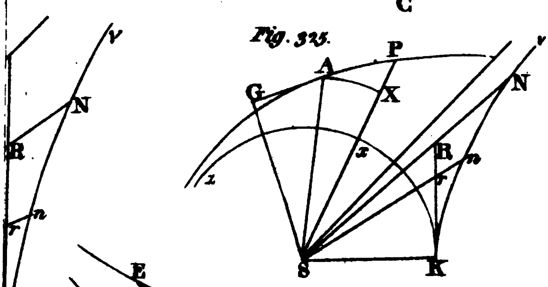


Fig. 325.



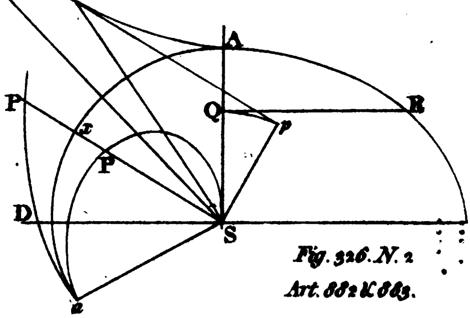


Fig. 326. N. 2

Art. 882 & 883.

llate XXXVIII. *To be plac'd at the End of Vol II.*

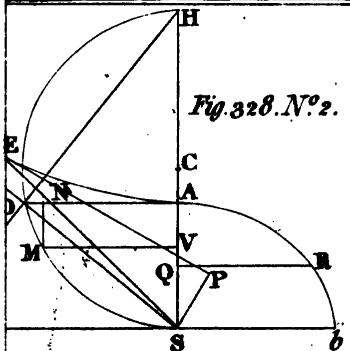
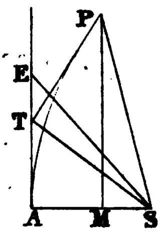


Fig. 328. N°. 2.

Fig. 329.



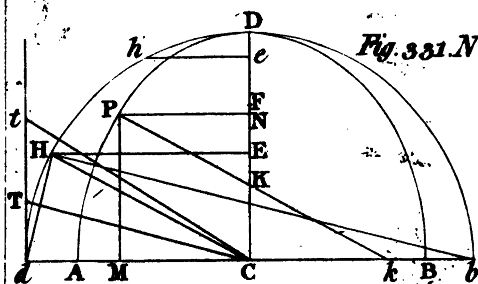


Fig. 331. N°. 1.

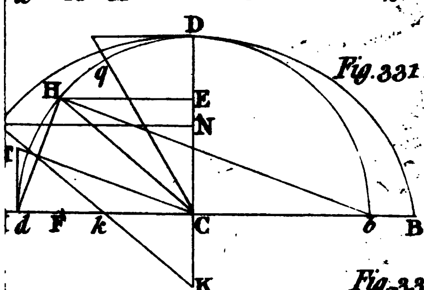


Fig. 331. N°. 2.

Fig. 334.

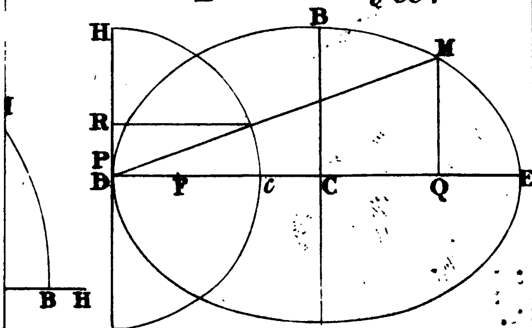


Plate XXXIX. *To be plac'd at the End of Vol. II.*

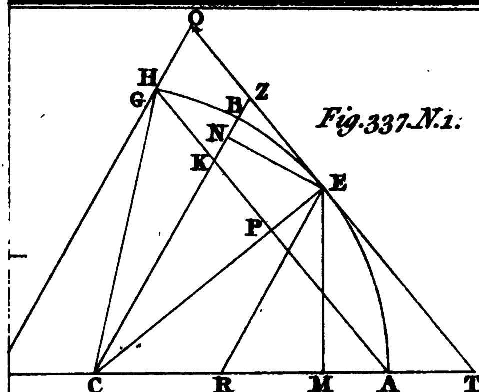


Fig. 337. N. 1.

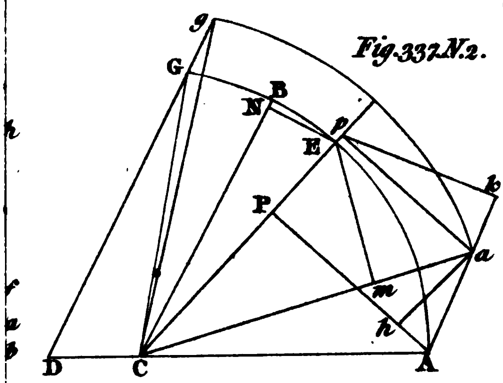
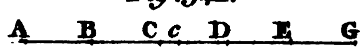


Fig. 337. N. 2.

Fig. 341.



[Fi]g. 340.

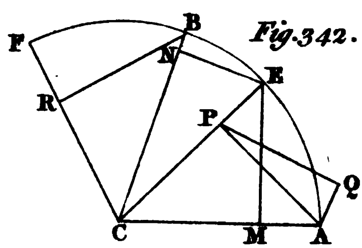
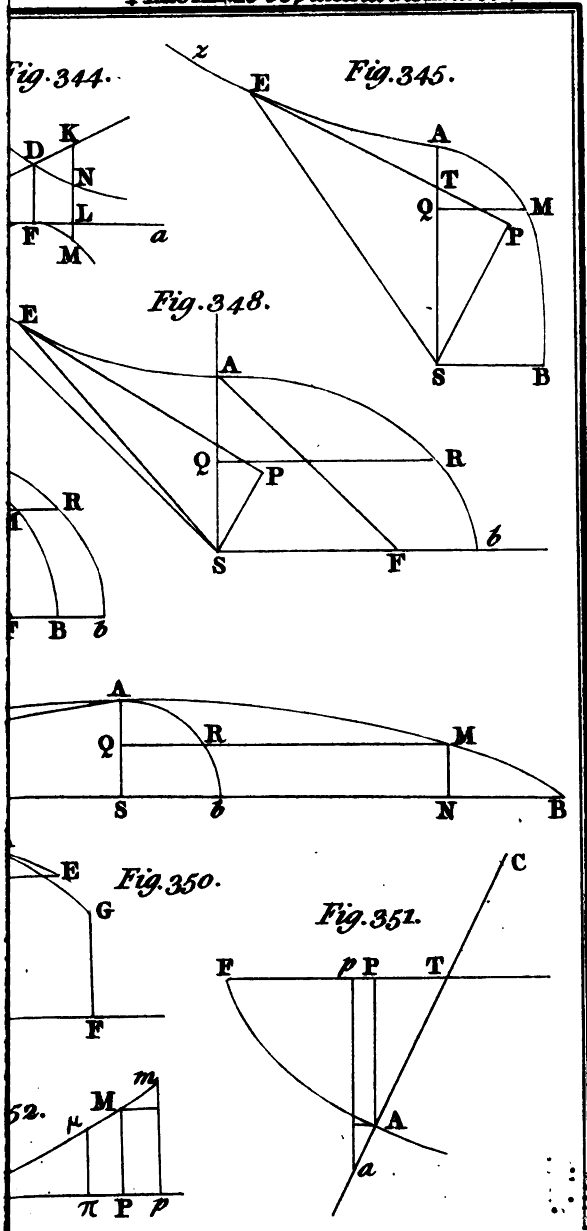


Fig. 342.

Plate XL. *To be plac'd at the End of Vol. II.*



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